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Non-holonomic Lagrangian and Hamiltonian mechanics: an intrinsic approach*

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Abstract

A geometrical approach to Lagrangian and Hamiltonian non-holonomic dynamics is proposed. The construction relies on a revisitation of the Poincaré–Cartan 1-form, leading to the introduction of the concepts of Lagrangian and Hamiltonian pairs and to the implementation of a non-holonomic Legendre map. The relationship with the standard 'extrinsic' approach is outlined. A unified 'canonical framework', joining both Lagrangian and Hamiltonian aspects, is proposed.

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1. Introduction

In recent years, the search for a deeper insight into the study of classical non-holonomic dynamics has brought about a great number of different geometric approaches [1, 3, 7–11, 13–20]. Despite the differences, all of them geometrize the presence of constraints by assigning the corresponding family of admissible kinetic states, regarded as a submanifold $\mathcal{A}$ of the velocity space.

In the present paper, attention will be paid to *Lagrangian systems*, meant as systems whose equations of motion are derivable from a suitable *Poincaré–Cartan* 1-form. The plan is to extend to the non-holonomic context the geometrical approach to classical Lagrangian and Hamiltonian mechanics developed in [4]. For the convenience of the reader, a brief review of the method is reported in section 2.

A major source of difficulty in the transition to non-holonomic systems comes from the fact that assigning a Lagrangian on the submanifold $\mathcal{A}$ is generally insufficient to reconstruct the constrained dynamics.

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The difficulty may be overcome by means of an 'extrinsic' algorithm, involving a Lagrangian defined over the whole velocity space, and the subsequent pull-back on $\mathcal{A}$ of the associated Poincaré–Cartan 1-form. However, as shown in appendix A, the simplification is only apparent, since the 'extrinsic' Lagrangian needed in order to reproduce the correct equations of motion on $\mathcal{A}$ is generally different from the *unconstrained* one, i.e. from the Lagrangian involved in the description of the evolution of the system as it would be *in the absence* of constraints.

For this reason, in section 3 we propose a totally intrinsic approach to the problem. The intrinsic Lagrangian and the Poincaré–Cartan 1-form are glued together into a *Lagrangian pair*, whose geometric properties allow a simple derivation of the Lagrange–Chetaev equations of motion. The section is ended by inspecting the relationship between the intrinsic set-up and the more traditional extrinsic one.

Finally, in section 4 we examine the relationship between Lagrangian pairs and *Legendre maps*, thereby opening the way to the *Hamiltonian* formulation of non-holonomic mechanics. In the resulting context, the concepts of *dynamical scheme* and *Hamiltonian pair* allow the building of a Hamiltonian scenario, perfectly symmetric to the Lagrangian one. Both scenarios are eventually joined into what we call a *canonical framework*, consisting of the submanifold $\mathcal{A}$ of admissible kinetic states, coupled with a partner submanifold on the Hamiltonian side, carrying dynamical information on the system. The interaction of the resulting set-up with the concepts of Lagrangian and Hamiltonian pair is thoroughly investigated.

The appendix provides a few explicit examples; apart from a practical explanation of the new methods, they are mainly aimed at giving evidence of the difference between 'extrinsic' and 'unconstrained' Lagrangians in the study of constrained dynamics.

2. Geometrical preliminaries

2.1. Generalities

For the convenience of the reader, we review here a few basic aspects of jet bundle geometry [2, 12], especially relevant to the subsequent discussion.

(i) Let $M \xrightarrow{t} \Re$ denote an $(m+1)$-dimensional differentiable manifold, fibred over the real line $\Re$, viewed as an Euclidean 1-space[1]. The first jet space $j_1(M) \xrightarrow{\pi} M$ is then an affine bundle over M, modelled on the vertical space $V(M)$, and identified with the submanifold of the tangent space $T(M)$ described by the equation

$$j_1(M) = \{z \mid z \in T(M), \langle z,\ \mathrm{d}t\rangle = 1\}. \tag{2.1}$$

The geometry of the manifold $j_1(M)$ will be regarded as known. For the notation, terminology, etc the reader is referred to [2] and references therein. Unless otherwise stated, we shall refer $j_1(M)$ to local jet coordinates $t, q^1, \ldots, q^m, \dot{q}^1, \ldots, \dot{q}^m$. The notation $\omega^i := \mathrm{d}q^i - \dot{q}^i\,\mathrm{d}t$ will be assumed throughout.

For each $z \in j_1(M)$, the vertical lift of vectors provides an isomorphism between the vertical spaces $V_{\pi(z)}(M)$ and $V_z(j_1(M))$, expressed in coordinates as

$$X^i \left(\frac{\partial}{\partial q^i}\right)_{\pi(z)} \longleftrightarrow X^i \left(\frac{\partial}{\partial \dot{q}^i}\right)_z . \tag{2.2a}$$

[1] A typical example is the configuration spacetime of a holonomic mechanical system, with the fibration $t : M \to \Re$ given by the absolute time function.

In a similar way, the projection $\pi : j_1(M) \to M$ allows every contact 1-form σ at z to be expressed as the pull-back of a 1-form $\hat{\sigma}$ at $\pi(z)$, thereby establishing a correspondence $C_z(j_1(M)) \to T^*_{\pi(z)}(M)$ summarized in the relation

$$\sigma_i \omega^i{}_{|z} \leftrightarrow \sigma_i [\mathrm{d}q^i - \dot{q}^i(z)\,\mathrm{d}t]_{\pi(z)}. \tag{2.2b}$$

Equations (2.2a) and (2.2b) determine a non-singular pairing $\langle \, \| \, \rangle$ between vertical vectors and contact 1-forms on $j_1(M)$, i.e. a *duality* between $V_z(j_1(M))$ and $C_z(j_1(M))$ $\forall z \in M$—based on the identification [2]

$$\left\langle \frac{\partial}{\partial \dot{q}^i} \,\middle\|\, \omega^j \right\rangle_z := \left\langle \frac{\partial}{\partial q^i}, \mathrm{d}q^j - \dot{q}^j(z)\,\mathrm{d}t \right\rangle_{\pi(z)} = \delta_i^j. \tag{2.3}$$

On the other hand, through ordinary pairing, every vector $X \in T_z(j_1(M))$ acts as a linear functional on $T^*_z(j_1(M))$, and therefore also on $C_z(j_1(M))$. For any $X \in T_z(j_1(M))$ there exists therefore a unique vector $J_z(X) \in V_z(j_1(M))$ satisfying the requirement

$$\langle J_z(X) \| \sigma \rangle = \langle X, \sigma \rangle \qquad \forall \sigma \in C_z(j_1(M)). \tag{2.4}$$

By varying z, the correspondence $X \to J_z(X)$ defines a tensor field J of type (1, 1) on $j_1(M)$, known as the *fundamental tensor*. In local coordinates equations (2.3) and (2.4) provide the explicit representation

$$J(X) = \langle J(X) \| \omega^i \rangle \frac{\partial}{\partial \dot{q}^i} = \langle X, \omega^i \rangle \frac{\partial}{\partial \dot{q}^i} \quad \Rightarrow \quad J = \frac{\partial}{\partial \dot{q}^i} \otimes \omega^i. \tag{2.5}$$

Strictly associated with J is an anti-derivation d_v of the Grassmann algebra over $j_1(M)$, known as the *fibre differential*, uniquely characterized by the requirements

$$\mathrm{d}_v f = \frac{\partial f}{\partial \dot{q}^k} \omega^k \qquad \mathrm{d}_v(\mathrm{d}f) + \mathrm{d}(\mathrm{d}_v f) + \mathrm{d}t \wedge \mathrm{d}f = 0 \qquad \forall f \in \mathcal{F}(j_1(M)). \tag{2.6}$$

An important property of the operator (2.6) is its *cohomological* character, expressed by the identity $\mathrm{d}_v \cdot \mathrm{d}_v \equiv 0$ [2].

(ii) Let $\mathcal{A}$ denote an embedded submanifold of $j_1(M)$, fibred over M. The situation, summarized in the commutative diagram

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\;i\;} & j_1(M) \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ M & = & M \end{array} \tag{2.7}$$

provides the natural setting for the study of dynamical systems in the presence of non-holonomic constraints [1, 3].

Referring $\mathcal{A}$ to local fibred coordinates $t, q^1, \ldots, q^m, z^1, \ldots, z^r$, the embedding $i : \mathcal{A} \to j_1(M)$ is expressed locally as

$$\dot{q}^i = \psi^i(t, q^1, \ldots, q^m, z^1, \ldots, z^r) \qquad i = 1, \ldots, m \tag{2.8a}$$

with $\operatorname{rank}\|\partial(\psi^1, \ldots, \psi^m)/\partial(z^1, \ldots, z^r)\| = r$. Alternatively, one may adopt the implicit representation

$$g^\sigma(t, q^1, \ldots, q^m, \dot{q}^1, \ldots, \dot{q}^m) = 0 \qquad \sigma = 1, \ldots, m - r \tag{2.8b}$$

with $\operatorname{rank}\|\partial(g^1, \ldots, g^{m-r})/\partial(\dot{q}^1, \ldots, \dot{q}^m)\| = m - r$.

For simplicity, in the following we shall not distinguish between the manifold $\mathcal{A}$ and its image $i(\mathcal{A}) \subset j_1(M)$. A section $\gamma : \Re \to M$ will be called $\mathcal{A}$-*admissible* (admissible, for short) if and only if its first jet extension is contained in $\mathcal{A}$.

Concepts such as dynamical flow, vertical vector and contact 1-form on $\mathcal{A}$ will be regarded as known [1, 3]. The vertical bundle and the contact bundle over $\mathcal{A}$ will be respectively denoted by $V(\mathcal{A})$ and $C(\mathcal{A})$. The notation $\tilde{\omega}^i := i^*(\omega^i) = \mathrm{d}q^i - \psi^i\,\mathrm{d}t$ will be used throughout.

(iii) The bilinear pairing (2.3) is easily adapted to the submanifold $\mathcal{A}$ through the identification

$$\left\langle \frac{\partial}{\partial z^A} \,\middle\|\, \tilde{\omega}^j \right\rangle_z := \left\langle i_*\left(\frac{\partial}{\partial z^A}\right) \,\middle\|\, \omega^j \right\rangle_{i(z)} = \left(\frac{\partial \psi^j}{\partial z^A}\right)_z$$

resulting in the representation

$$\langle V \| \nu \rangle = V^A \nu_j \frac{\partial \psi^j}{\partial z^A} \qquad \forall V = V^A \frac{\partial}{\partial z^A} \quad \nu = \nu_j \tilde{\omega}^j. \tag{2.9}$$

Of course, the correspondence $V(\mathcal{A}) \times C(\mathcal{A}) \to \mathcal{F}(\mathcal{A})$ expressed by equation (2.9) has now a *singular* character; it is clear that any 1-form $\nu = \nu_i \tilde{\omega}^i$ satisfying $\nu_i \frac{\partial \psi^i}{\partial z^A} = 0$, $A = 1, \ldots, r$ annihilates all vertical vectors.

The totality of such 1-forms generates a vector sub-bundle $\chi(\mathcal{A}) \subset C(\mathcal{A})$ of the contact bundle, known as the *Chetaev bundle* [3]. Every section $\nu : \mathcal{A} \to \chi(\mathcal{A})$ is called a *Chetaev 1-form* on $\mathcal{A}$.

A local basis for the Chetaev bundle consists of any set of linearly independent contact 1-forms $\mu^\sigma = \mu^\sigma{}_i \tilde{\omega}^i$, $\sigma = 1, \ldots, m-r$ satisfying the conditions

$$\mu^\sigma{}_i \frac{\partial \psi^i}{\partial z^A} = 0 \qquad \sigma = 1, \ldots, m-r \quad A = 1, \ldots, r.$$

In particular, given any implicit representation (2.8*b*) for the submanifold $\mathcal{A}$, a possible choice is provided by $\mu^\sigma = i^*(\mathrm{d}_v g^\sigma) = i^*\left(\frac{\partial g^\sigma}{\partial \dot{q}^i}\right)\tilde{\omega}^i$.

Remark 2.1. Due to its singular character, the pairing (2.9) does not determine a duality between $V(\mathcal{A})$ and $C(\mathcal{A})$, thereby preventing any direct implementation of the algorithm involved in the definition of the fundamental tensor and of the associated *fibre differential*. As we shall see, this plays a crucial role in the transition from holonomic to non-holonomic Lagrangian mechanics.

2.2. The Lagrangian bundles

In recent papers [4, 6], a new geometrical setting for a gauge-invariant formulation of Lagrangian mechanics has been exploited. For the convenience of the reader, we outline here the main features of the method.

(i) With every mechanical system $\mathcal{B}$ subject to (smooth) positional constraints, we associate a double fibration $P \xrightarrow{\pi} \mathcal{V}_{n+1} \xrightarrow{t} \Re$, where

- (i) $\mathcal{V}_{n+1} \xrightarrow{t} \Re$ is the configuration spacetime of $\mathcal{B}$, endowed with the absolute time function;
- (ii) $P \xrightarrow{\pi} \mathcal{V}_{n+1}$ is a principal fibre bundle with structural group $(\Re, +)$, diffeomorphic, in a non-canonical way, to the Cartesian product $\mathcal{V}_{n+1} \times \Re$, called the bundle of *affine scalars* over $\mathcal{V}_{n+1}$.

The action of $(\Re, +)$ on P results in a 1-parameter group of diffeomorphisms $\psi_\xi : P \to P$, expressed through the additive notation

$$(\nu, \xi) \in P \times \Re \to \psi_\xi(\nu) := \nu + \xi \in P. \tag{2.10}$$

Every function $u : P \to \Re$ satisfying $u(\nu + \xi) = u(\nu) + \xi$ is called a *trivialization* of P.

The assignment of a trivialization u allows the lifting of every local coordinate system $t, q^1, \ldots, q^n$ over $\mathcal{V}_{n+1}$ to a corresponding 'fibred' coordinate system $t, q^1, \ldots, q^n, u$ over P. The group of fibred coordinate transformations has the form

$$\bar{t} = t + c \qquad \bar{q}^i = \bar{q}^i(t, q^1, \ldots, q^n) \qquad \bar{u} = u + f(t, q^1, \ldots, q^n).$$

(ii) The (pull-back of the) absolute time function provides a fibration $P \xrightarrow{t} \Re$. Let $j_1(P, \Re) \xrightarrow{\pi} P$ denote the associated first-jet space. As usual, we shall refer $j_1(P, \Re)$ to local jet-coordinates $t, q^i, u, \dot{q}^i, \dot{u}$, with transformation laws

$$\bar{t} = t + c \qquad \bar{q}^i = \bar{q}^i(t, q) \qquad \bar{u} = u + f(t, q) \tag{2.11a}$$

$$\dot{\bar{q}}^i = \frac{\partial \bar{q}^i}{\partial q^k}\dot{q}^k + \frac{\partial \bar{q}^i}{\partial t} \qquad \dot{\bar{u}} = \dot{u} + \frac{\partial f}{\partial q^k}\dot{q}^k + \frac{\partial f}{\partial t} := \dot{u} + \dot{f}. \tag{2.11b}$$

The geometrical properties of $j_1(P, \Re)$ include, in the first place, all the attributes coming from the jet-bundle structure (fundamental tensor, fibre differential, etc). A local basis for the contact bundle $C(j_1(P, \Re))$, dual of the basis $\frac{\partial}{\partial \dot{u}}, \frac{\partial}{\partial \dot{q}^i}$ of the vertical bundle $V(j_1(P, \Re))$ under the pairing $\langle \, \| \, \rangle$, is provided by the 1-forms,

$$\omega^0 = \mathrm{d}u - \dot{u}\,\mathrm{d}t \qquad \omega^k = \mathrm{d}q^k - \dot{q}^k\,\mathrm{d}t \qquad k = 1, \ldots, n. \tag{2.12}$$

In addition to the jet attributes, the space $j_1(P, \Re)$ inherits from P two distinguished actions of the group $(\Re, +)$, expressed in coordinates as

$$\psi_{\xi *} : (t, q^i, u, \dot{q}^i, \dot{u}) \longrightarrow (t, q^i, u + \xi, \dot{q}^i, \dot{u}) \tag{2.13a}$$

$$\phi_{\xi} : (t, q^i, u, \dot{q}^i, \dot{u}) \longrightarrow (t, q^i, u, \dot{q}^i, \dot{u} + \xi). \tag{2.13b}$$

Referring to [4] for the necessary details, we focus on the following basic facts:

- The direct product of the actions (2.13*a*) and (2.13*b*) makes $j_1(P, \Re)$ into a principal fibre bundle over $j_1(\mathcal{V}_{n+1})$, with projection $j_1(P, \Re) \xrightarrow{\pi_2} j_1(\mathcal{V}_{n+1})$ expressed locally as
$$\pi_2 : (t, q^i, u, \dot{q}^i, \dot{u}) \longrightarrow (t, q^i, \dot{q}^i). \tag{2.14}$$
- The quotient of $j_1(P, \Re)$ by the action (2.13*a*), denoted by $\mathcal{L}(\mathcal{V}_{n+1})$, is a $(2n + 2)$-dimensional manifold, with coordinates $t, q^i, \dot{q}^i, \dot{u}$, called the *Lagrangian bundle*; the quotient map makes $j_1(P, \Re)$ into a principal fibre bundle over $\mathcal{L}(\mathcal{V}_{n+1})$, with structural group $(\Re, +)$; at the same time, the action (2.13*b*) 'passes to the quotient', making $\mathcal{L}(\mathcal{V}_{n+1})$ into a principal fibre bundle over $j_1(\mathcal{V}_{n+1})$, again with structural group $(\Re, +)$.
- In a similar way, the quotient of $j_1(P, \Re)$ by the action (2.13*b*), denoted by $\mathcal{L}^c(\mathcal{V}_{n+1})$, is a $(2n + 2)$-dimensional manifold, with coordinates $t, q^i, u, \dot{q}^i$, called the *co-Lagrangian bundle*; the quotient map makes $j_1(P, \Re)$ into a principal fibre bundle over $\mathcal{L}^c(\mathcal{V}_{n+1})$, with structural group $(\Re, +)$, while the action (2.13*a*), transferred to the quotient space, makes $\mathcal{L}^c(\mathcal{V}_{n+1})$ into a principal fibre bundle over $j_1(\mathcal{V}_{n+1})$, with structural group $(\Re, +)$.

The situation, summarized into the diagram

$$\begin{tikzcd} j_1(P, \Re) \arrow[r] \arrow[d] & \mathcal{L}^c(\mathcal{V}_{n+1}) \arrow[d] \\ \mathcal{L}(\mathcal{V}_{n+1}) \arrow[r] & j_1(\mathcal{V}_{n+1}) \end{tikzcd} \tag{2.15}$$

provides the necessary tool for a gauge-invariant formulation of Lagrangian mechanics. As outlined in [4], this is achieved by giving up the traditional approach, based on the interpretation of the Lagrangian function $L(t, q^i, \dot{q}^i)$ as the representation of a (gauge-dependent) *scalar field* over $j_1(\mathcal{V}_{n+1})$, and introducing instead the concept of *Lagrangian section*, meant as a section $l : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$ of the Lagrangian bundle.

For each choice of the trivialization u of P, the description of l takes the local form

$$\dot{u} = L(t, q^i, \dot{q}^i) \tag{2.16}$$

i.e. it does still rely on the assignment of a function $L(t, q^i, \dot{q}^i)$ over $j_1(\mathcal{V}_{n+1})$. However, as soon as the trivialization is changed into $\bar{u} = u + f$, the representation (2.16) undergoes the transformation law

$$\dot{\bar{u}} = \dot{u} + \dot{f} = L(t, q^i, \dot{q}^i) + \dot{f} := L'(t, q^i, \dot{q}^i)$$

involving a different, gauge-equivalent 'Lagrangian'.

Referring to [4] for further comments, we concentrate on the algorithm assigning to each Lagrangian section l a corresponding *Poincaré–Cartan* 1*-form* on $j_1(\mathcal{V}_{n+1})$. To this end, starting with l, we consider in turn:

- the trivialization ψ_l of the bundle $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ induced by l, described locally by the function $\dot{u} - L(t, q^i, \dot{q}^i)$;
- the pull-back of ψ_l on $j_1(P, \Re)$, denoted by $\hat{\psi}_l$, and described locally by the function $\hat{\psi}_l(t, q^i, u, \dot{q}^i, \dot{u}) = \dot{u} - L(t, q^i, \dot{q}^i)$.

It is then an easy matter to verify the validity of the following assertions:

(a) $\hat{\psi}_l$ is a trivialization of the bundle $j_1(P, \Re) \to \mathcal{L}^c(\mathcal{V}_{n+1})$; as such, it determines a section $\hat{l} : \mathcal{L}^c(\mathcal{V}_{n+1}) \to j_1(P, \Re)$, expressed locally as $\dot{u} = L(t, q^i, \dot{q}^i)$; the sections l and $\hat{l}$, together, provide a principal bundle homomorphism, summarized in the commutative diagram

$$\begin{tikzcd} \mathcal{L}^c(\mathcal{V}_{n+1}) \arrow[r, "\hat{l}"] \arrow[d] & j_1(P, \Re) \arrow[d] \\ j_1(\mathcal{V}_{n+1}) \arrow[r, "l"] & \mathcal{L}(\mathcal{V}_{n+1}) \end{tikzcd} \tag{2.17}$$

(b) The fibre differential $\mathrm{d}_v \hat{\psi}_l$, expressed locally as

$$\mathrm{d}_v \hat{\psi}_l = \mathrm{d}_v[\dot{u} - L(t, q^i, \dot{q}^i)] = \omega^0 - \frac{\partial L}{\partial \dot{q}^k}\omega^k \tag{2.18}$$

determines a *connection* on the principal bundle $j_1(P, \Re) \to \mathcal{L}(\mathcal{V}_{n+1})$.

In view of (a) and (b), the pull-back of $\mathrm{d}_v \hat{\psi}_l$ through the diagram (2.17), defines a connection on $\mathcal{L}^c(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$, described locally by the 1-form

$$\hat{l}^*(\mathrm{d}_v \hat{\psi}_l) = \mathrm{d}u - L\,\mathrm{d}t - \frac{\partial L}{\partial \dot{q}^k}\omega^k. \tag{2.19}$$

The difference $\mathrm{d}u - \hat{l}^*(\mathrm{d}_v \hat{\psi}_l)$ is then (the pull-back of) a 1-form ϑ over $j_1(\mathcal{V}_{n+1})$, called the *Poincaré–Cartan* 1*-form* of l, expressed in coordinates as

$$\vartheta = L\,\mathrm{d}t + \frac{\partial L}{\partial \dot{q}^k}\omega^k. \tag{2.20}$$

In other words, for each choice of the trivialization u of P, the Poincaré–Cartan 1-form is nothing but a representation of the connection on the bundle $\mathcal{L}^c(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$ induced by the Lagrangian section l through the process indicated above. The exterior differential $\Omega := \mathrm{d}\vartheta$, known as the *Poincaré–Cartan* 2*-form*, is therefore a gauge-invariant object over $j_1(\mathcal{V}_{n+1})$, identical, up to a sign, to the *curvature* of the connection (2.19).

By means of the correspondence $l \to \Omega$ we recover the entire content of classical Lagrangian mechanics. The argument is well known. For further information, the reader is referred to [4] and references therein.

Remark 2.2. In view of equation (2.20), interior multiplication by Ω converts *vertical* vectors into *contact* 1-*forms*, thereby giving rise to a linear map $g : V(j_1(\mathcal{V}_{n+1})) \to C(j_1(\mathcal{V}_{n+1}))$, expressed in coordinates as

$$X = X^i \frac{\partial}{\partial \dot{q}^i} \quad \Rightarrow \quad g(X) := X \rfloor \Omega = \mathfrak{L}_X(\vartheta) = X^i \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^k} \omega^k. \tag{2.21}$$

Coupled with the pairing (2.3), the correspondence (2.21) determines a symmetric scalar product on $V(j_1(\mathcal{V}_{n+1}))$, based on the prescription

$$(X, Y) = \langle X \| g(Y) \rangle = \frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j} X^i Y^j. \tag{2.22}$$

The section l is called *regular* if and only if the scalar product (2.22) is non-degenerate, i.e. if and only if the matrix $\frac{\partial^2 L}{\partial \dot{q}^i \partial \dot{q}^j}$ is everywhere non-singular.

2.3. The Hamiltonian bundles

Paralleling the discussion in section 2.2, we now consider the fibration $P \to \mathcal{V}_{n+1}$, and denote by $\pi : j_1(P, \mathcal{V}_{n+1}) \to P$ the associated first jet space.

Every fibred coordinate system t, q^i, u on P induces local coordinates t, q^i, u, p_0, p_i on $j_1(P, \mathcal{V}_{n+1})$, with transformation group

$$\bar{t} = t + c \qquad \bar{q}^i = \bar{q}^i(t, q^1, \ldots, q^n), \qquad \bar{u} = u + f(t, q^1, \ldots, q^n) \tag{2.23a}$$

$$\bar{p}_0 = p_0 + \frac{\partial f}{\partial t} + \left(p_k + \frac{\partial f}{\partial q^k}\right)\frac{\partial q^k}{\partial \bar{t}} \qquad \bar{p}_i = \left(p_k + \frac{\partial f}{\partial q^k}\right)\frac{\partial q^k}{\partial \bar{q}^i}. \tag{2.23b}$$

Equations (2.23*a*) and (2.23*b*) ensure the invariance of the contact 1-form

$$\Theta = \mathrm{d}u - p_0\,\mathrm{d}t - p_i\,\mathrm{d}q^i \tag{2.24}$$

known as the *Liouville* 1-*form* of $j_1(P, \mathcal{V}_{n+1})$.

Exactly as in the Lagrangian case, one can easily establish two distinguished actions of the group $(\Re, +)$ over $j_1(P, \mathcal{V}_{n+1})$, expressed synthetically as

$$\psi_{\xi*} : (t, q^i, u, p_0, p_i) \longrightarrow (t, q^i, u + \xi, p_0, p_i) \tag{2.25a}$$

$$\phi_\xi : (t, q^i, u, p_0, p_i) \quad \longrightarrow \quad (t, q^i, u, p_0 + \xi, p_i). \tag{2.25b}$$

Referring to [4] for the necessary details, we focus on the following facts:

- The direct product of the actions (2.25*a*) and (2.25*b*) makes $j_1(P, \mathcal{V}_{n+1})$ into a principal fibre bundle over a $(2n + 1)$-dimensional base space $\Pi(\mathcal{V}_{n+1})$ with coordinates t, q^i, p_i, called the *phase space*.
- The quotient of $j_1(P, \mathcal{V}_{n+1})$ by the action (2.25*a*), denoted by $\mathcal{H}(\mathcal{V}_{n+1})$, is an affine bundle over $\mathcal{V}_{n+1}$, with coordinates t, q^i, p_0, p_i, modelled on the cotangent space $T^*(\mathcal{V}_{n+1})$ and called the *Hamiltonian bundle*; the quotient map makes $j_1(P, \mathcal{V}_{n+1})$ into a principal fibre bundle over $\mathcal{H}(\mathcal{V}_{n+1})$, with structural group $(\Re, +)$; the canonical 1-form (2.24) endows $j_1(P, \mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ with a distinguished connection, called the *canonical connection*; at the same time, the action (2.25*b*) 'passes to the quotient', making $\mathcal{H}(\mathcal{V}_{n+1})$ into a principal fibre bundle over the phase space $\Pi(\mathcal{V}_{n+1})$.

- The quotient of $j_1(P, \mathcal{V}_{n+1})$ by the action (2.25*b*), denoted by $\mathcal{H}^c(\mathcal{V}_{n+1})$, is a $(2n+2)$-dimensional manifold, with coordinates t, q^i, u, p_i, called the *co-Hamiltonian bundle*; the quotient map makes $j_1(P, \mathcal{V}_{n+1})$ into a principal fibre bundle over $\mathcal{H}^c(\mathcal{V}_{n+1})$; at the same time, the action (2.25*a*), transferred to $\mathcal{H}^c(\mathcal{V}_{n+1})$, makes the latter into a principal fibre bundle over $\Pi(\mathcal{V}_{n+1})$.

The situation, summarized into the commutative diagram

$$\begin{tikzcd} j_1(P, \mathcal{V}_{n+1}) \arrow[r] \arrow[d] & \mathcal{H}^c(\mathcal{V}_{n+1}) \arrow[d] \\ \mathcal{H}(\mathcal{V}_{n+1}) \arrow[r] & \Pi(\mathcal{V}_{n+1}) \end{tikzcd} \tag{2.26}$$

provides the starting point for a gauge-invariant formulation of Hamiltonian mechanics [5].

3. Non-holonomic Lagrangian dynamics

3.1. Non-holonomic Lagrangian bundles

Let us return to the diagram (2.7), with the base manifold M now explicitly identified with the configuration spacetime $\mathcal{V}_{n+1}$ of a material system $\mathcal{B}$, and with the embedding $i : \mathcal{A} \to j_1(\mathcal{V}_{n+1})$ taken as a description of the *kinetic constraints* acting on $\mathcal{B}$ [1, 3]. The construction of the Lagrangian bundles is easily adapted to the submanifold $\mathcal{A}$, through a straightforward pull-back procedure.

As usual, the situation is conveniently expressed through a commutative diagram

$$\begin{tikzcd} & j_1(P, \Re) \arrow[rr] \arrow[dd] & & \mathcal{L}^c(\mathcal{V}_{n+1}) \arrow[dd] \\ j_1^{\mathcal{A}}(P, \Re) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathcal{L}^c(\mathcal{A}) \arrow[ur] \arrow[dd] & \\ & \mathcal{L}(\mathcal{V}_{n+1}) \arrow[rr] & & j_1(\mathcal{V}_{n+1}) \\ \mathcal{L}(\mathcal{A}) \arrow[ur] \arrow[rr] & & \mathcal{A} \arrow[ur] & \end{tikzcd} \tag{3.1}$$

in which all arrows are implicitly defined by the context (namely, $\mathcal{L}(\mathcal{A})$ and $\mathcal{L}^c(\mathcal{A})$ are respectively the pull-back of $\mathcal{L}(\mathcal{V}_{n+1})$ and $\mathcal{L}^c(\mathcal{V}_{n+1})$ on the submanifold $\mathcal{A} \to j_1(\mathcal{V}_{n+1})$, $j_1^{\mathcal{A}}(P, \Re)$ is alternatively the pull-back of $j_1(P, \Re) \to \mathcal{L}(\mathcal{V}_{n+1})$ on the submanifold $\mathcal{L}(\mathcal{A}) \to \mathcal{L}(\mathcal{V}_{n+1})$, or the pull-back of $j_1(P, \Re) \to \mathcal{L}^c(\mathcal{V}_{n+1})$ on $\mathcal{L}^c(\mathcal{A}) \to \mathcal{L}^c(\mathcal{V}_{n+1})$, etc).

As usual, we refer $\mathcal{A}$ to local fibred coordinates $t, q^1, \ldots, q^n, z^1, \ldots, z^r$, with transformation laws

$$\bar{t} = t + c \qquad \bar{q}^i = \bar{q}^i(t, q^k) \qquad \bar{z}^A = \bar{z}^A(t, q^k, z^B) \tag{3.2}$$

and express the embedding $i : \mathcal{A} \to j_1(\mathcal{V}_{n+1})$ in the form

$$\dot{q}^i = \psi^i(t, q^k, z^A). \tag{3.3}$$

The following assertions are then entirely straightforward:

- Every choice of a trivialization u of P allows the lifting of the coordinates on $\mathcal{A}$ to local coordinates $t, q^i, z^A, \dot{u}$ on $\mathcal{L}(\mathcal{A})$, t, q^i, u, z^A on $\mathcal{L}^c(\mathcal{A})$ and $t, q^i, u, z^A, \dot{u}$ on $j_1^{\mathcal{A}}(P, \Re)$; the resulting coordinate transformations are obtained by completing equations (3.2) with (the significant part of) the system

$$\bar{u} = u + f(t, q^i) \qquad \bar{\dot{u}} = \dot{u} + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^k}\psi^k(t, q^i, z^A) := \dot{u} + \dot{f}_{|\mathcal{A}}.$$

- All the embeddings $\mathcal{L}(\mathcal{A}) \to \mathcal{L}(\mathcal{V}_{n+1})$, $\mathcal{L}^c(\mathcal{A}) \to \mathcal{L}^c(\mathcal{V}_{n+1})$, $j_1^{\mathcal{A}}(P, \Re) \to j_1(P, \Re)$ are described locally by equation (3.3).
- A local basis for the contact bundle $C\big(j_1^{\mathcal{A}}(P, \Re)\big)$ is provided by the 1-forms

$$\tilde{\omega}^0 = \mathrm{d}u - \dot{u}\,\mathrm{d}t \qquad \tilde{\omega}^i = \mathrm{d}q^i - \psi^i\,\mathrm{d}t. \tag{3.4}$$

- The restriction to the submanifold $j_1^{\mathcal{A}}(P, \Re)$ of the projection (2.14) gives rise to a surjection $j_1^{\mathcal{A}}(P, \Re) \to \mathcal{A}$, still denoted by π_2, and expressed in coordinates as

$$\pi_2 : (t, q^i, u, z^A, \dot{u}) \longrightarrow (t, q^i, z^A). \tag{3.5}$$

- Both actions (2.13*a*) and (2.13*b*) of the group $(\Re, +)$ on $j_1(P, \Re)$ preserve the submanifold $j_1^{\mathcal{A}}(P, \Re)$, thereby inducing corresponding actions $(\psi_\xi)_* : j_1^{\mathcal{A}}(P, \Re) \to j_1^{\mathcal{A}}(P, \Re)$ and $\phi_\xi : j_1^{\mathcal{A}}(P, \Re) \to j_1^{\mathcal{A}}(P, \Re)$, expressed in coordinates as

$$\psi_{\xi *} : (t, q^i, u, z^A, \dot{u}) \longrightarrow (t, q^i, u + \xi, z^A, \dot{u})$$
$$\phi_\xi : (t, q^i, u, z^A, \dot{u}) \longrightarrow (t, q^i, u, z^A, \dot{u} + \xi).$$

  From this, proceeding as in section 2.2, it is easily seen that the manifold $j_1^{\mathcal{A}}(P, \Re)$ is a principal fibre bundle over $\mathcal{L}(\mathcal{A})$ under the action $(\psi_\xi)_*$, as well as a principal fibre bundle over $\mathcal{L}^c(\mathcal{A})$ under the action ϕ_ξ, and that both $\mathcal{L}(\mathcal{A})$ and $\mathcal{L}^c(\mathcal{A})$ are principal fibre bundles over $\mathcal{A}$ under the (induced) actions of $(\psi_\xi)_*$ and ϕ_ξ respectively. Accordingly, all arrows in the front and rear faces of the diagram (3.1) express principal fibrations, while those in the left and right faces are principal bundle homomorphisms.

Preserving the terminology of section 2.2, the principal fibre bundles $\mathcal{L}(\mathcal{A}) \to \mathcal{A}$ and $\mathcal{L}^c(\mathcal{A}) \to \mathcal{A}$ will be respectively called the non-holonomic Lagrangian and co-Lagrangian bundle over $\mathcal{A}$. A section $l : \mathcal{A} \to \mathcal{L}(\mathcal{A})$ will be called a (non-holonomic) Lagrangian section. Once a trivialization u of P has been fixed, the local description of any such section takes the explicit form,

$$\dot{u} = L(t, q^i, z^A). \tag{3.6}$$

The function on the right-hand side of equation (3.6) will be called a *Lagrangian* on $\mathcal{A}$. Under an arbitrary change $u \to u + f$ of the trivialization, the representation (3.6) undergoes the transformation law,

$$\bar{\dot{u}} = \dot{u} + \dot{f}_{|\mathcal{A}} = L + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^i}\psi^i. \tag{3.7}$$

Let us now explore the possibility of extending to the non-holonomic context the algorithm assigning to every Lagrangian section (3.6) a corresponding Poincaré–Cartan 1-form on $\mathcal{A}$. To this end, proceeding as in section 2.2, we convert l into a trivialization $\psi_l := \dot{u} - L(t, q^i, z^A)$ of the bundle $\mathcal{L}(\mathcal{A}) \to \mathcal{A}$, pull the result back on $j_1^{\mathcal{A}}(P, \Re)$, thus getting a trivialization $\hat{\psi}_l = \dot{u} - L(t, q^i, z^A)$ of the bundle $j_1^{\mathcal{A}}(P, \Re) \to \mathcal{L}^c(\mathcal{A})$, and use the latter to induce a section $\hat{l} : \mathcal{L}^c(\mathcal{A}) \to j_1^{\mathcal{A}}(P, \Re)$, with local equation $\dot{u} = L(t, q^i, z^A)$.

In this way, we get the non-holonomic analogue of the bundle homomorphism (2.17), namely

$$
\begin{array}{ccc}
\mathcal{L}^c(\mathcal{A}) & \xrightarrow{\hat{l}} & j_1^{\mathcal{A}}(P,\mathfrak{R}) \\
\Big\downarrow & & \Big\downarrow \\
\mathcal{A} & \xrightarrow{l} & \mathcal{L}(\mathcal{A})
\end{array}
\quad . \tag{3.8}
$$

Unfortunately, the analogy ends here: the subsequent step, namely the creation of a connection 1-form over the bundle $j_1^{\mathcal{A}}(P,\mathfrak{R}) \to \mathcal{L}(\mathcal{A})$ through fibre differentiation of the function $\hat{\psi}_l$, is in fact precluded by the fact that the geometrical attributes of $j_1^{\mathcal{A}}(P,\mathfrak{R})$ do not include any such operation.

The difficulty may be overcome in various ways. A commonly adopted procedure consists in modifying the starting point, assigning, instead of l, an 'extrinsic' section $\tilde{l} : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$, defined globally on $j_1(\mathcal{V}_{n+1})$, or, at least, on an open neighbourhood of the submanifold $\mathcal{A}$. One may then implement the algorithm of section 2.2 up to the construction of the Poincaré–Cartan 1-form on $j_1(\mathcal{V}_{n+1})$, pull the result back on $\mathcal{A}$, and call it the non-holonomic Poincaré–Cartan 1-form of the system. The method is perfectly all right, as long as $\tilde{l}$ is consistently regarded as an attribute of the *constrained* system, not to be confused with any sort of 'free' Lagrangian associated with the fictitious system resulting from the given one by removing the kinetic constraints. A few examples illustrating this aspect may be found in appendix A.

Of course, one may choose to *define* the extrinsic Lagrangian section $\tilde{l}$ as the one (if any!) yielding the correct Lagrange–Chetaev equations of motion on $\mathcal{A}$. This is a legitimate viewpoint, essentially equivalent to a restatement of the inverse problem in the presence of kinetic constraints. The advantages of such an approach, however, are more apparent than real, since, in general, the determination of $\tilde{l}$ depends not only on the active interactions but also, explicitly, on the nature of the reactive forces. In this respect, a genuinely intrinsic approach seems more appropriate.

In what follows, we propose a detailed analysis of this point. Throughout the discussion, the main emphasis will be put on the equations of motion, and in particular, on the construction of a non-holonomic analogue of the Poincaré–Cartan formalism. The relationship between the intrinsic viewpoint and the constitutive characterization of the constraints will be examined in a forthcoming paper.

3.2. Lagrangian pairs

Pursuing the viewpoint initiated in section 3.1, we now focus on the front face of diagram (3.1), namely

$$
\begin{array}{ccc}
j_1^{\mathcal{A}}(P,\mathfrak{R}) & \longrightarrow & \mathcal{L}^c(\mathcal{A}) \\
\Big\downarrow & & \Big\downarrow \\
\mathcal{L}(\mathcal{A}) & \longrightarrow & \mathcal{A}
\end{array}
\tag{3.9}
$$

and regard it as the natural set-up for the construction of an intrinsic, non-holonomic Lagrangian formalism. Let $C(\mathcal{A})$ and $C\big(j_1^{\mathcal{A}}(P,\mathfrak{R})\big)$ denote the contact bundles on the manifolds $\mathcal{A}$ and $j_1^{\mathcal{A}}(P,\mathfrak{R})$ respectively.

Definition 3.1. *A connection on the principal fibre bundle* $j_1^{\mathcal{A}}(P,\mathfrak{R}) \to \mathcal{L}(\mathcal{A})$ *is called a* contact connection *if and only if*

- *the associated connection* 1*-form* $\hat{\vartheta}$ *is a contact* 1*-form over* $j_1^A(P,\Re)$*;*
- *for each choice of the trivialization* u*, the difference* $\tilde{\omega}^0 - \hat{\vartheta}$ *is the pull-back of a contact* 1*-form over* $\mathcal{A}$ *with respect to the projection (3.5).*

According to definition 3.1, every contact connection 1-form admits a local representation of the form

$$\hat{\vartheta} = \tilde{\omega}^0 - \varphi_i\,\tilde{\omega}^i = \mathrm{d}u - \dot{u}\,\mathrm{d}t - \varphi_i(\mathrm{d}q^i - \psi^i\,\mathrm{d}t) \tag{3.10}$$

with $\varphi_i = \varphi_i(t, q^i, z^A) \in \mathcal{F}(\mathcal{A})$. Under an arbitrary change of trivialization $\bar{u} = u + f$, equation (3.10) is transformed into

$$\hat{\vartheta} = \mathrm{d}\bar{u} - \dot{\bar{u}}\,\mathrm{d}t - (\mathrm{d}f - \dot{f}\,\mathrm{d}t) - \varphi_i\tilde{\omega}^i = \overline{\tilde{\omega}}^0 - \overline{\varphi}_i\tilde{\omega}^i$$

with $\overline{\varphi}_i = \varphi_i + \frac{\partial f}{\partial q^i}$. In view of the stated properties, the contact connections over $j_1^A(P,\Re) \to \mathcal{L}(\mathcal{A})$ are easily recognized to form an affine bundle over $\mathcal{A}$, modelled on the contact bundle $C(\mathcal{A})$, and diffeomorphic to the pull-back of the phase space $\Pi(\mathcal{V}_{n+1}) \to \mathcal{V}_{n+1}$ over $\mathcal{A} \to \mathcal{V}_{n+1}$.

Let us now envisage a geometrical set-up consisting in the simultaneous assignment of a Lagrangian section $l : \mathcal{A} \to \mathcal{L}(\mathcal{A})$, expressed locally as $\dot{u} = L(t, q^i, z^A)$, and of a contact connection on $j_1^A(P,\Re)$, with connection 1-form (3.10). In this way, proceeding as in section 3.1, namely lifting l to a section $\hat{l} : \mathcal{L}^c(\mathcal{A}) \to j_1^A(P,\Re)$, and pulling back $\hat{\vartheta}$ through $\hat{l}$, we end up with a connection on the principal fibre bundle $\mathcal{L}^c(\mathcal{A}) \to \mathcal{A}$, described by the connection 1-form,

$$\hat{l}^*(\hat{\vartheta}) = \mathrm{d}u - L(t, q^i, z^A)\,\mathrm{d}t - \varphi_i(t, q^i, z^A)\tilde{\omega}^i. \tag{3.11}$$

This is precisely what was done in section 2.2, the (essential!) difference is that, in the holonomic context, the contact connection $\hat{\vartheta}$ was not an independent piece of information, but was uniquely determined by the knowledge of l.

The plan is to generalize this state of affairs, expressing the relationship between l and $\hat{\vartheta}$ in an implicit form, suitable for arbitrary (not necessarily holonomic) systems. To this end, we resort once again to the fact that the difference

$$\vartheta := \mathrm{d}u - \hat{l}^*(\hat{\vartheta}) = L\,\mathrm{d}t + \varphi_i\tilde{\omega}^i \tag{3.12}$$

is (the pull-back of) a 1-form over $\mathcal{A}$, and consider the exterior differential $\Omega = \mathrm{d}\vartheta$, identical, up to a sign, to the *curvature* of the connection 1-form $\hat{l}^*(\hat{\vartheta})$.

Let $X \to X \,\lrcorner\, \Omega$ denote the linear endomorphism between vectors and 1-forms induced by Ω on $\mathcal{A}$. For *vertical* vectors $X = X^A \frac{\partial}{\partial z^A}$, a straightforward comparison with equation (3.12) provides the explicit representation,

$$X \,\lrcorner\, \Omega = \mathfrak{L}_X(\vartheta) = X^A \left[\left(\frac{\partial L}{\partial z^A} - \varphi_i \frac{\partial \psi^i}{\partial z^A} \right) \mathrm{d}t + \frac{\partial \varphi_i}{\partial z^A}\tilde{\omega}^i \right]. \tag{3.13}$$

From this we derive the following basic conclusions:

- A necessary and sufficient condition for the correspondence (3.13) to map vertical vectors into contact 1-forms is that the functions $L(t, q^i, z^A)$ and $\varphi_i(t, q^i, z^A)$ involved in the local representation of the section l and of the connection $\hat{\vartheta}$ satisfy the consistency relations,

$$\frac{\partial L}{\partial z^A} = \varphi_i \frac{\partial \psi^i}{\partial z^A} \qquad A = 1, \ldots, r. \tag{3.14}$$

  Under the stated circumstance, l and $\hat{\vartheta}$ will be said to form a *Lagrangian pair* over $\mathcal{A}$. The 1-form (3.12) will be called the Poincaré–Cartan 1-form of $(l, \hat{\vartheta})$, and the exterior differential $\Omega = \mathrm{d}\vartheta$ the associated Poincaré–Cartan 2-form.

Note that, in view of the identity

$$\frac{\partial \dot f}{\partial z^A} = \frac{\partial}{\partial z^A}\left(\frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^i}\psi^i\right) = \frac{\partial f}{\partial q^i}\frac{\partial \psi^i}{\partial z^A}$$

the requirement (3.14) is *invariant* under arbitrary gauge transformations $u \to u + f$ $\left(\Rightarrow L \to L + \dot f, \varphi_i \to \varphi_i + \frac{\partial f}{\partial q^i}\right)$;

- Given any Lagrangian pair $(l, \hat\vartheta)$, the restriction of the map $X \to X \,\lrcorner\, \Omega$ to the vertical bundle will be denoted by $g : V(\mathcal{A}) \to C(\mathcal{A})$. Comparison with equation (3.13) yields the explicit expression

$$g\left(\frac{\partial}{\partial z^A}\right) = \frac{\partial \varphi_i}{\partial z^A}\tilde\omega^i \qquad A = 1, \ldots, r. \tag{3.15}$$

Through equation (3.15), every Lagrangian pair $(l, \hat\vartheta)$ determines a (possibly singular) scalar product between vertical vectors, based on the prescription (see equations (2.9), (2.22) and (3.15))

$$(X, Y) := \langle X \| g(Y)\rangle = X^A Y^B \frac{\partial \varphi_i}{\partial z^B}\frac{\partial \psi^i}{\partial z^A}. \tag{3.16}$$

Introducing the notation

$$G_{AB} := \left(\frac{\partial}{\partial z^A}, \frac{\partial}{\partial z^B}\right) = \frac{\partial \varphi_i}{\partial z^B}\frac{\partial \psi^i}{\partial z^A} \tag{3.17}$$

equation (3.14) yields the symmetry relation

$$G_{AB} = \frac{\partial^2 L}{\partial z^A \partial z^B} - \varphi_i \frac{\partial^2 \psi^i}{\partial z^A \partial z^B} = G_{BA} \tag{3.18}$$

mathematically equivalent to $(X, Y) = (Y, X)$.

Note that, by construction, the scalar product (3.16) depends on the 1-form $\hat\vartheta$, but is independent of the section l. Conversely, we can state

Proposition 3.1. *A contact connection $\hat\vartheta$ may be completed (locally) to a Lagrangian pair $(l, \hat\vartheta)$ if and only if the functions $\varphi_i(t, q^i, z^A)$ involved in the representation (3.10) satisfy the symmetry requirement*

$$\frac{\partial \varphi_i}{\partial z^A}\frac{\partial \psi^i}{\partial z^B} = \frac{\partial \varphi_i}{\partial z^B}\frac{\partial \psi^i}{\partial z^A} \qquad \forall A, B = 1, \ldots, r. \tag{3.19}$$

Under the stated assumption, the section l is determined up to an arbitrary transformation $l \to l + \varrho(t, q^i)$.

Proof. Equations (3.19) are mathematically equivalent to the conditions

$$\frac{\partial}{\partial z^A}\left(\varphi_i \frac{\partial \psi^i}{\partial z^B}\right) - \frac{\partial}{\partial z^B}\left(\varphi_i \frac{\partial \psi^i}{\partial z^A}\right) = 0$$

i.e. to the local solvability of the system

$$\frac{\partial L}{\partial z^A} = \varphi_i \frac{\partial \psi^i}{\partial z^A}$$

with the 'unknown' $L = L(t, q^i, z^A)$ determined up to an arbitrary function $\varrho(t, q^i)$. The result is invariant under arbitrary transformations $\varphi_i \to \varphi_i + \frac{\partial f}{\partial q^i}$, provided that L is transformed into $L + \frac{\mathrm{d}f}{\mathrm{d}t}$. All assertions of proposition 3.1 follow easily from this fact. □

By analogy with the holonomic case, a Lagrangian pair $(l, \hat{\vartheta})$ will be called *regular* if and only if the associated scalar product (3.16) is non-degenerate, i.e. if and only if the matrix (3.17) is everywhere non-singular on $\mathcal{A}$. More generally, a contact connection $\hat{\vartheta}$ will be called *regular* if and only if it may be completed locally to a regular Lagrangian pair.

Under the regularity assumption, recalling the definition of the Chetaev bundle $\chi(\mathcal{A})$, it is an easy matter to verify that, at each $z \in \mathcal{A}$, the image of the vertical space $V_z(\mathcal{A})$ under the map (3.15) is an r-dimensional subspace $g(V_z(\mathcal{A})) \subset C_z(\mathcal{A})$ satisfying the relation[2]

$$g(V_z(\mathcal{A})) \cap \chi_z(\mathcal{A}) = \{0\}. \tag{3.20}$$

By varying z we conclude

Proposition 3.2. *Every regular contact connection determines a splitting of the contact bundle $C(\mathcal{A})$ into the direct sum*

$$C(\mathcal{A}) = g(V(\mathcal{A})) \oplus_{\mathcal{A}} \chi(\mathcal{A}). \tag{3.21}$$

The relevance of proposition 3.2 in the study of the reactive forces will be discussed in a forthcoming paper. At present, we shall concentrate on the relationship between Lagrangian pairs and *dynamical flows* on $\mathcal{A}$. To this end, given any Lagrangian pair $(l, \hat{\vartheta})$, we consider the ideal of exterior differential forms on $\mathcal{A}$ generated by the Poincaré–Cartan 2-form Ω of $(l, \hat{\vartheta})$ and by the Chetaev 1-forms, and denote by $\mathcal{D}(l, \hat{\vartheta})$ ($\mathcal{D}$ for short) the associated characteristic distribution,

$$\mathcal{D} := \{X \mid X \in T(\mathcal{A}),\, X \,\lrcorner\, \Omega \in \chi(\mathcal{A}),\, X \,\lrcorner\, \chi(\mathcal{A}) = \{0\}\}. \tag{3.22}$$

We have then the following:

Theorem 3.1. *Given any regular Lagrangian pair $(l, \hat{\vartheta})$ on $\mathcal{A}$, described locally by equations (3.6) and (3.10), the associated characteristic distribution (3.22) coincides with the linear span of a dynamical flow Z, uniquely determined by the Lagrange–Chetaev equations*

$$\left(Z(\varphi_i) - \frac{\partial L}{\partial q^i} + \varphi_k \frac{\partial \psi^k}{\partial q^i}\right)\frac{\partial \psi^i}{\partial z^A} = 0. \tag{3.23}$$

Proof. A straightforward check shows that the most general vector X satisfying $X \,\lrcorner\, \nu = 0, \forall \nu \in \chi(\mathcal{A})$ is necessarily of the form

$$X = X^0\left(\frac{\partial}{\partial t} + \psi^i \frac{\partial}{\partial q^i}\right) + X^A \frac{\partial \psi^i}{\partial z^A}\frac{\partial}{\partial q^i} + V^A \frac{\partial}{\partial z^A} \tag{3.24}$$

for arbitrary choices of X^0, X^A and V^A. At the same time, taking equations (3.12) and (3.14) into account, the Poincaré–Cartan 2-form of $(l, \hat{\vartheta})$ is easily recognized to have the local expression

$$\Omega = \mathrm{d}(L\,\mathrm{d}t + \varphi_i \tilde{\omega}^i) = \left[\left(\varphi_k \frac{\partial \psi^k}{\partial q^i} - \frac{\partial L}{\partial q^i}\right)\mathrm{d}t + \mathrm{d}\varphi_i\right] \wedge \tilde{\omega}^i. \tag{3.25}$$

In view of equation (3.25), the characteristic distribution (3.22) consists of the totality of vectors of the form (3.24) satisfying the requirement

$$\left(\varphi_k \frac{\partial \psi^k}{\partial q^i} - \frac{\partial L}{\partial q^i}\right)\left(X^0 \tilde{\omega}^i - X^A \frac{\partial \psi^i}{\partial z^A}\,\mathrm{d}t\right) + \langle X, \mathrm{d}\varphi_i\rangle \tilde{\omega}^i - X^A \frac{\partial \psi^i}{\partial z^A}\,\mathrm{d}\varphi_i \in \chi(\mathcal{A}).$$

[2] Indeed for any $X \in V_z(\mathcal{A})$, the condition $\langle g(X), Y\rangle = 0, \forall Y \in V_z(\mathcal{A})$ implies $X = 0$, thus establishing in one step both the *injectivity* of g and equation (3.20).

Using the expansion

$$\mathrm{d}\varphi_i = \left(\frac{\partial\varphi_i}{\partial t} + \frac{\partial\varphi_i}{\partial q^k}\psi^k\right)\mathrm{d}t + \frac{\partial\varphi_i}{\partial q^k}\tilde{\omega}^k + \frac{\partial\varphi_i}{\partial z^B}\,\mathrm{d}z^B$$

and recalling the definition of $\chi(\mathcal{A})$, the latter condition splits into the system

$$X^A\frac{\partial\psi^i}{\partial z^A}\left(\varphi_k\frac{\partial\psi^k}{\partial q^i} - \frac{\partial L}{\partial q^i} + \frac{\partial\varphi_i}{\partial t} + \frac{\partial\varphi_i}{\partial q^k}\psi^k\right) = 0 \qquad X^A\frac{\partial\psi^i}{\partial z^A}\frac{\partial\varphi_i}{\partial z^B} = 0$$

$$\left[X^0\left(\varphi_k\frac{\partial\psi^k}{\partial q^i} - \frac{\partial L}{\partial q^i} + \frac{\partial\varphi_i}{\partial t} + \frac{\partial\varphi_i}{\partial q^k}\psi^k\right) + X^A\frac{\partial\psi^k}{\partial z^A}\left(\frac{\partial\varphi_i}{\partial q^k} - \frac{\partial\varphi_k}{\partial q^i}\right) + V^A\frac{\partial\varphi_i}{\partial z^A}\right]\frac{\partial\psi^i}{\partial z^B} = 0.$$

From this, taking definition (3.17) into account, it is easily seen that, under the regularity assumption $\det G_{AB} \neq 0$, the characteristic distribution (3.22) coincides with the linear module spanned by the vector field

$$Z = \frac{\partial}{\partial t} + \psi^i\frac{\partial}{\partial q^i} + Z^A\frac{\partial}{\partial z^A} \tag{3.26a}$$

with components Z^A uniquely determined by the system

$$\left(\varphi_k\frac{\partial\psi^k}{\partial q^i} - \frac{\partial L}{\partial q^i} + \frac{\partial\varphi_i}{\partial t} + \psi^k\frac{\partial\varphi_i}{\partial q^k} + Z^A\frac{\partial\varphi_i}{\partial z^A}\right)\frac{\partial\psi^i}{\partial z^B} = 0. \tag{3.26b}$$

Equation (3.26*a*) shows that Z is a dynamical flow over $\mathcal{A}$, while equations (3.26*b*) reproduce the content of the Lagrange–Chetaev equations (3.23). □

Equivalent representations of equations (3.23) are

$$Z\left(\frac{\partial L}{\partial z^A}\right) - \frac{\partial L}{\partial q^i}\frac{\partial\psi^i}{\partial z^A} = \varphi_k\left[Z\left(\frac{\partial\psi^k}{\partial z^A}\right) - \frac{\partial\psi^k}{\partial q^i}\frac{\partial\psi^i}{\partial z^A}\right]$$

obtained by comparison with equation (3.14), and especially useful in the presence of *linear* kinetic constraints, and

$$Z(\varphi_i) - \frac{\partial L}{\partial q^i} + \varphi_k\frac{\partial\psi^k}{\partial q^i} = \lambda_\sigma i^*\left(\frac{\partial g^\sigma}{\partial\dot{q}^i}\right)$$

relying on the extrinsic representation (2.8*b*) for the submanifold $\mathcal{A}$, and consisting of n independent equations, for the simultaneous determination of the dynamical flow Z, and the unknown multipliers $\lambda_\sigma(t, q^i, \dot{q}^i)$.

In any case, the important point is that, independently of the specific formulation, every regular Lagrangian pair $(l, \hat{\vartheta})$ gives rise to a well-posed problem of motion through the associated characteristic distribution (3.22). In view of equations (3.26*a*) and (3.26*b*) the equations of motion take the explicit form

$$\begin{cases} \dfrac{\mathrm{d}q^i}{\mathrm{d}t} = \psi^i(t, q^i, z^A) \\ \dfrac{\mathrm{d}z^A}{\mathrm{d}t} = Z^A(t, q^i, z^A) = -G^{AB}\left(\varphi_k\dfrac{\partial\psi^k}{\partial q^i} - \dfrac{\partial L}{\partial q^i} + \dfrac{\partial\varphi_i}{\partial t} + \dfrac{\partial\varphi_i}{\partial q^k}\psi^k\right)\dfrac{\partial\psi^i}{\partial z^B} \end{cases}$$

where G^{AB} denotes the inverse matrix of G_{AB}.

3.3. Extrinsic formulation

Although not explicitly relevant for the subsequent discussion, it is worth spending a few words on the relationship between *Lagrangian pairs* and *extrinsic Lagrangian sections*. To this end, we resume the notation of section 3.1, and denote generically by i any of the maps joining the front ('non-holonomic') face of diagram (3.1) to the rear ('holonomic') one, and by i^* the associated pull-back[3].

Given an extrinsic Lagrangian section $\tilde{l} : j_1(\mathcal{V}_{n+1}) \to \mathcal{L}(\mathcal{V}_{n+1})$, expressed locally as $\dot{u} = \tilde{L}(t, q^i, \dot{q}^i)$, let $\mathrm{d}_v\hat{\psi}_{\tilde{l}}$ denote the connection on $j_1(P, \Re) \to \mathcal{L}(\mathcal{V}_{n+1})$ induced by $\tilde{l}$ through the algorithm (2.18). It is then an easy matter to verify that $(i^*(\tilde{l}), i^*(\mathrm{d}_v\hat{\psi}_{\tilde{l}}))$ is automatically a Lagrangian pair on $\mathcal{A}$, and that, whenever $\tilde{l}$ is regular, so is $(i^*(\tilde{l}), i^*(\mathrm{d}_v\hat{\psi}_{\tilde{l}}))$. In other words, (regular) extrinsic Lagrangian sections induce (regular) Lagrangian pairs.

We shall now establish a local converse of this result, namely that every (regular) Lagrangian pair is generated locally by a (regular) extrinsic Lagrangian section. To this end, we start with an elementary result of linear algebra.

Lemma 3.1. *Consider any square matrix of the form*

$$\Lambda = \begin{pmatrix} A & B \\ {}^tB & C \end{pmatrix}.$$

Then, under the assumption $\det A \neq 0$*, a necessary and sufficient condition for the non-singularity of* Λ *is the non-singularity of the matrix* $C - {}^tBA^{-1}B$.

Proof. The kernel of Λ consists of all vectors $X = \binom{U}{V}$ satisfying

$$\Lambda(X) = \begin{pmatrix} AU + BV \\ {}^tBU + CV \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Whenever the matrix A is non-singular, this is equivalent to the pair of conditions

$$U = -A^{-1}BV \qquad (-{}^tBA^{-1}B + C)V = 0$$

whence the result. □

Theorem 3.2. *Let* $(l, \hat{\vartheta})$ *be any (regular) Lagrangian pair on* $\mathcal{A}$*. Then, for each* $z \in \mathcal{A}$ *there exists a neighbourhood* U *of* $i(z)$ *in* $j_1(\mathcal{V}_{n+1})$*, and a (regular) section* $\tilde{l} : U \to \mathcal{L}(\mathcal{V}_{n+1})$ *satisfying* $i^*(\tilde{l}) = l$ *and* $i^*(\mathrm{d}_v\hat{\psi}_{\tilde{l}}) = \hat{\vartheta}$ *everywhere on* $U \cap \mathcal{A}$.

Proof. Let $g^\sigma(t, q^i, \dot{q}^i) = 0, \sigma = 1, \dots, n - r$ denote any Cartesian representation of $\mathcal{A}$. Extend the coordinates z^A on $\mathcal{A}$ to functions $z^A(t, q^i, \dot{q}^i)$ on $j_1(\mathcal{V}_{n+1})$. The variables t, q^i, z^A, g^σ are then independent in a neighbourhood U of each point $i(z), z \in \mathcal{A}$, thus allowing the $\dot{q}^i$ to be expressed in the form $\dot{q}^i = \dot{q}^i(t, q^i, z^A, g^\sigma)$. Of course, by definition, we have the identities $i^*(g^\sigma) = i^*(\dot{q}^i - \psi^i) \equiv 0$. Given any trivialization u of P, let $\dot{u} = L(t, q^i, z^A)$ and $\hat{\vartheta} = \tilde{\omega}^0 + \varphi_i(t, q^i, z^A)\tilde{\omega}^i$ denote the corresponding local representation of the pair $(l, \hat{\vartheta})$. Define a function $\tilde{L}$ on U on the basis of the requirement

$$\tilde{L}(t, q^i, \dot{q}^i) = L + \varphi_k(\dot{q}^k - \psi^k) + \tfrac{1}{2}Q_{\alpha\beta}g^\alpha g^\beta \tag{3.27}$$

where $Q_{\alpha\beta}(t, q^i, \dot{q}^i)$ denotes a (so far) arbitrary matrix function. Under an arbitrary change of trivialization $u \to u + f$, equation (3.27) implies the transformation law

$$\tilde{L} \to \tilde{L} + \frac{\partial f}{\partial t} + \psi^k\frac{\partial f}{\partial q^k} + \frac{\partial f}{\partial q^k}(\dot{q}^k - \psi^k) = \tilde{L} + \frac{\mathrm{d}f}{\mathrm{d}t}$$

[3] For example, $i^*(\tilde{l}) := \tilde{l} \cdot i$ will denote the restriction of the section $\tilde{l}$ to the submanifold $\mathcal{A}$, $i^*(\mathrm{d}_v\hat{\psi}_{\tilde{l}})$ the pull-back of the 1-form $\mathrm{d}_v\hat{\psi}_{\tilde{l}}$ to the submanifold $j_1^{\mathcal{A}}(P, \Re) \subset j_1(P, \Re)$, etc.

showing that the position $\dot{u} = \tilde{L}(t, q^i, \dot{q}^i)$ does indeed define a section $\tilde{l}$ of the bundle $\mathcal{L}(\mathcal{V}_{n+1}) \to j_1(\mathcal{V}_{n+1})$. By equation (3.27), taking the consistency conditions (3.14) into account, we derive the relations

$$\frac{\partial \tilde{L}}{\partial \dot{q}^j} = \frac{\partial \varphi_k}{\partial z^B}\frac{\partial z^B}{\partial \dot{q}^j}(\dot{q}^k - \psi^k) + \varphi_j + \frac{1}{2}\frac{\partial Q_{\alpha\beta}}{\partial \dot{q}^j} g^\alpha g^\beta + Q_{\alpha\beta} g^\alpha \frac{\partial g^\beta}{\partial \dot{q}^j}. \tag{3.28}$$

Now:

- independently of the choice of the matrix $Q_{\alpha\beta}$, equations (3.27) and (3.28) provide the identifications
$$i^*(\tilde{L}) = L \qquad i^*\left(\frac{\partial \tilde{L}}{\partial \dot{q}^j}\right) = \varphi_j$$
mathematically equivalent to $i^*(\tilde{l}) = l$, $i^*(\mathrm{d}_v \hat{\psi}_{\tilde{l}}) = \hat{\vartheta}$;
- at each point $z \in \mathcal{A}$, recalling definition (3.17) as well as the relation $i^*(\dot{q}^i) = \psi^i$, equations (3.28) yield the representation
$$\frac{\partial^2 \tilde{L}}{\partial \dot{q}^i \partial \dot{q}^j} = \frac{\partial \varphi_j}{\partial z^A}\frac{\partial z^A}{\partial \dot{q}^i} + \frac{\partial \varphi_i}{\partial z^A}\frac{\partial z^A}{\partial \dot{q}^j} - G_{AB}\frac{\partial z^A}{\partial \dot{q}^i}\frac{\partial z^B}{\partial \dot{q}^j} + Q_{\alpha\beta}\frac{\partial g^\alpha}{\partial \dot{q}^h}\frac{\partial g^\beta}{\partial \dot{q}^k}.$$

From this, it is an easy matter to verify that the non-singularity of $\frac{\partial^2 \tilde{L}}{\partial \dot{q}^i \partial \dot{q}^j}$ at z is mathematically equivalent to the non-singularity of the matrix

$$\begin{pmatrix} \dfrac{\partial^2 \tilde{L}}{\partial \dot{q}^i \partial \dot{q}^j}\dfrac{\partial \dot{q}^i}{\partial z^A}\dfrac{\partial \dot{q}^j}{\partial z^B} & \dfrac{\partial^2 \tilde{L}}{\partial \dot{q}^i \partial \dot{q}^j}\dfrac{\partial \dot{q}^i}{\partial z^A}\dfrac{\partial \dot{q}^j}{\partial g^\beta} \\ \dfrac{\partial^2 \tilde{L}}{\partial \dot{q}^i \partial \dot{q}^j}\dfrac{\partial \dot{q}^i}{\partial g^\alpha}\dfrac{\partial \dot{q}^j}{\partial z^B} & \dfrac{\partial^2 \tilde{L}}{\partial \dot{q}^i \partial \dot{q}^j}\dfrac{\partial \dot{q}^i}{\partial g^\alpha}\dfrac{\partial \dot{q}^j}{\partial g^\beta} \end{pmatrix} = \begin{pmatrix} G & N \\ {}^tN & Q \end{pmatrix}$$

with $G = G_{AB}$, $Q = Q_{\alpha\beta}$ and $N = N_{A\beta} := \frac{\partial \varphi_i}{\partial z^A}\frac{\partial \dot{q}^i}{\partial g^\beta}$.

In particular, under the regularity assumption $\det G \neq 0$, lemma 3.1 shows that any choice of the matrix $Q_{\alpha\beta}$ satisfying the condition $\det(Q - {}^tNG^{-1}N) \neq 0$ is sufficient to ensure the regularity of the local section $\tilde{l}$, associated with the given Lagrangian pair. □

Theorem 3.2 legitimates the extrinsic approach to the equations of motion outlined in section 3.1. The analysis helps to clarify that, in general, the determination of an extrinsic Lagrangian section $\tilde{l}$ yielding the correct equations of motion on $\mathcal{A}$ has nothing to do with the description of the dynamical behaviour of the unconstrained system.

Corollary 3.1. *Let $(l, \hat{\vartheta})$ and $\tilde{l}$ denote respectively a non-singular Lagrangian pair and a non-singular Lagrangian section, related by the pull-back algorithm described in theorem 3.2. Then, the dynamical flow Z induced by $(l, \hat{\vartheta})$ on $\mathcal{A}$ is i-related to the dynamical flow $\tilde{Z}$ on $j_1(\mathcal{V}_{n+1})$ implicitly defined by the $2n - r$ equations,*

$$\tilde{Z}\left(\frac{\partial \tilde{L}}{\partial \dot{q}^k}\right) - \frac{\partial \tilde{L}}{\partial q^k} = \lambda_\sigma \frac{\partial g^\sigma}{\partial \dot{q}^k} \tag{3.29a}$$

$$\tilde{Z}(g^\sigma) = 0. \tag{3.29b}$$

Proof. Owing to the assumed regularity of $\tilde{l}$, equations (3.29*a*) and (3.29*b*) determine a unique dynamical flow $\tilde{Z}$ on $j_1(\mathcal{V}_{n+1})$. The latter is automatically tangent to $\mathcal{A}$, and *i*-related to a dynamical flow Z on $\mathcal{A}$, uniquely determined by the conditions

$$Z\left[i^*\left(\frac{\partial \tilde{L}}{\partial \dot{q}^k}\right)\right] = i^*\left(\frac{\partial \tilde{L}}{\partial q^k} + \lambda_\sigma \frac{\partial g^\sigma}{\partial \dot{q}^k}\right). \tag{3.30}$$

Comparison with equations (3.27) and (3.28) provides the identifications

$$i^*\left(\frac{\partial \tilde{L}}{\partial \dot{q}^k}\right) = \varphi_k \qquad i^*\left(\frac{\partial \tilde{L}}{\partial q^k}\right) = \frac{\partial L}{\partial q^k} - \varphi_i \frac{\partial \psi^i}{\partial q^k}.$$

Equations (3.30) are therefore identical to the Lagrange–Chetaev equations (3.23) involved in the determination of the dynamical flow induced by the pair $(l, \hat{\vartheta})$ on $\mathcal{A}$. □

4. Hamiltonian formulation

4.1. The Legendre map

The intrinsic Lagrangian approach outlined in section 3.2 has a natural Hamiltonian counterpart, based on a non-holonomic analogue of the Legendre transformation. Exactly as in the holonomic case [4], the algorithm relies on the fact that the attributes of the principal fibre bundle $j_1(P, \mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{V}_{n+1})$ include a geometrically distinguished *canonical connection* Θ, expressed in coordinates as

$$\Theta = \mathrm{d}u - p_0\,\mathrm{d}t - p_i\,\mathrm{d}q^i. \tag{4.1}$$

On this basis, taking the results of section 3.2 into account, we state

Proposition 4.1. *Every regular contact connection $\hat{\vartheta}$ determines a bundle map*

$$\begin{tikzcd} j_1^{\mathcal{A}}(P, \Re) \arrow[r, "\Lambda"] \arrow[d] & j_1(P, \mathcal{V}_{n+1}) \arrow[d] \\ P \arrow[r, equal] & P \end{tikzcd} \tag{4.2}$$

fibred over P, and satisfying the condition

$$\Lambda^*(\Theta) = \hat{\vartheta}. \tag{4.3}$$

The correspondence (4.2) induces an overall map of the non-holonomic Lagrangian bundles into the (holonomic) Hamiltonian ones, summarized in the commutative diagram

$$\begin{tikzcd} & j_1(P, \mathcal{V}_{n+1}) \arrow[rr] \arrow[dd] & & \mathcal{H}^c(\mathcal{V}_{n+1}) \arrow[dd] \\ j_1^{\mathcal{A}}(P, \Re) \arrow[ur, "\Lambda"] \arrow[rr] \arrow[dd] & & \mathcal{L}^c(\mathcal{A}) \arrow[ur, "\widehat{\Lambda}"] \arrow[dd] & \\ & \mathcal{H}(\mathcal{V}_{n+1}) \arrow[rr] & & \Pi(\mathcal{V}_{n+1}) \\ \mathcal{L}(\mathcal{A}) \arrow[ur, "\widetilde{\Lambda}"] \arrow[rr] & & \mathcal{A} \arrow[ur, "\lambda"] & \end{tikzcd} \tag{4.4}$$

where

- *the front and rear faces reproduce diagrams (3.9) and (2.26);*
- *each pair of arrows joining an edge on the front (Lagrangian) face with an edge on the rear (Hamiltonian) one represents a principal bundle homomorphism.*

If $\hat{\vartheta}$ is a regular *contact connection, all maps from the Lagrangian to the Hamiltonian side of diagram (4.4) are immersions.*

Proof. In view of equations (3.10) and (4.1), the requirement (4.3) singles out a unique differentiable map $\Lambda : j_1^{\mathcal{A}}(P, \mathfrak{R}) \to j_1(P, \mathcal{V}_{n+1})$, expressed in coordinates as

$$t = t \qquad q^i = q^i \qquad u = u \qquad p_0 = \dot{u} - \varphi_i \psi^i \qquad p_i = \varphi_i(t, q^i, z^A). \tag{4.5}$$

By equation (3.17) it is also seen that, whenever $\hat{\vartheta}$ is a regular contact connection, the Jacobian $\frac{\partial(p_1 \cdots p_n)}{\partial(z^1 \cdots z^r)}$ has rank r everywhere on $\mathcal{A}$. All assertions of proposition 4.1 are straightforward consequences of these facts. □

For dynamical purposes, let us now concentrate on the lower face of diagram (4.4), namely

$$\begin{array}{ccc} \mathcal{L}(\mathcal{A}) & \xrightarrow{\widetilde{\Lambda}} & \mathcal{H}(\mathcal{V}_{n+1}) \\ \big\downarrow & & \big\downarrow \\ \mathcal{A} & \xrightarrow{\lambda} & \Pi(\mathcal{V}_{n+1}) \end{array} . \tag{4.6}$$

By analogy with the holonomic case, the correspondence $\lambda : \mathcal{A} \to \Pi(\mathcal{V}_{n+1})$ will be called the (non-holonomic) *Legendre map* associated with the connection $\hat{\vartheta}$. Comparison with equation (4.5) provides the local representation

$$p_i = \varphi_i(t, q^i, z^A) \tag{4.7}$$

showing that knowledge of λ is equivalent to knowledge of $\hat{\vartheta}$.

Definition 4.1. *A regular contact connection $\hat{\vartheta}$ will be called hyperregular if and only if the associated Legendre map is an embedding, i.e. if and only if the image space $\lambda(\mathcal{A})$ is a submanifold of $\Pi(\mathcal{V}_{n+1})$. A regular Lagrangian pair $(l, \hat{\vartheta})$ will be called hyperregular if and only if $\hat{\vartheta}$ is a hyperregular connection.*

Remark 4.1. Assuming the simultaneous validity of both circumstances:

- $\mathcal{A}$ is an affine subbundle of $j_1(\mathcal{V}_{n+1})$ ($\Leftrightarrow$ the kinetic constraints are *linear* in the velocities);
- $\lambda : \mathcal{A} \to \Pi(\mathcal{V}_{n+1})$ is an affine map ($\Leftrightarrow$ the p_i depend linearly on the velocities);

it is easily seen that every regular connection is automatically hyperregular.

Remark 4.2. Exactly as happened in the Lagrangian context, given any submanifold $i : \mathcal{S} \to \Pi(\mathcal{V}_{n+1})$ fibred over $\mathcal{V}_{n+1}$, the Hamiltonian bundles (2.26) may be pulled back to $\mathcal{S}$. The resulting structure, summarized in the diagram

$$\begin{tikzcd}
 & j_1(P, \mathcal{V}_{n+1}) \arrow[rr] \arrow[dd] & & \mathcal{H}^c(\mathcal{V}_{n+1}) \arrow[dd] \\
j_1^{\mathcal{S}}(P, \mathcal{V}_{n+1}) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathcal{H}^c(\mathcal{S}) \arrow[ur] \arrow[dd] & \\
 & \mathcal{H}(\mathcal{V}_{n+1}) \arrow[rr] & & \Pi(\mathcal{V}_{n+1}) \\
\mathcal{H}(\mathcal{S}) \arrow[ur] \arrow[rr] & & \mathcal{S} \arrow[ur] &
\end{tikzcd} \tag{4.8}$$

may be regarded as a sort of 'Hamiltonian analogue' of the situation described in section 3.1. Consistently with this viewpoint, the principal fibrations $\mathcal{H}(\mathcal{S}) \to \mathcal{S}$ and $\mathcal{H}^c(\mathcal{S}) \to \mathcal{S}$

will be respectively called the non-holonomic *Hamiltonian bundle* and the non-holonomic *co-Hamiltonian bundle* over $\mathcal{S}$.

The pull-back of the Liouville 1-form (2.24) through the upper-left arrow of diagram (4.8) defines a 1-form $\hat{\Theta}$ on $j_1^{\mathcal{S}}(P, \mathcal{V}_{n+1})$, henceforth called the *non-holonomic Liouville* 1*-form*. Exactly as in section 2.3, it is easily recognized that $\hat{\Theta}$ has the nature of a *connection* 1*-form* over the principal bundle $j_1^{\mathcal{S}}(P, \mathcal{V}_{n+1}) \to \mathcal{H}(\mathcal{S})$. The exterior differential $\hat{\Omega} := -\mathrm{d}\hat{\Theta}$, identical, up to a sign, to the *curvature* of the Liouville connection, is therefore a geometrical object over the base manifold $\mathcal{H}(\mathcal{S})$. As we shall see, this fact plays an important role in the Hamiltonian formulation of non-holonomic mechanics.

The previous arguments are especially relevant in connection with definition 4.1. In fact, given any hyperregular contact connection $\hat{\vartheta}$, let $\mathcal{S} := \lambda(\mathcal{A})$ denote the submanifold of $\Pi(\mathcal{V}_{n+1})$ determined by the Legendre map (4.7). Taking both proposition 4.1 and remark 4.2 into account, it is then an easy matter to verify that diagram (4.4) gives rise to an overall isomorphism

$$
\begin{tikzcd}
 & j_1^{\mathcal{S}}(P, \mathcal{V}_{n+1}) \arrow[rr] \arrow[dd] & & \mathcal{H}^c(\mathcal{S}) \arrow[dd] \\
j_1^{\mathcal{A}}(P, \Re) \arrow[ur] \arrow[rr] \arrow[dd] & & \mathcal{L}^c(\mathcal{V}_{n+1}) \arrow[ur] \arrow[dd] & \\
 & \mathcal{H}(\mathcal{S}) \arrow[rr] & & \mathcal{S} \\
\mathcal{L}(\mathcal{A}) \arrow[ur] \arrow[rr] & & \mathcal{A} \arrow[ur] &
\end{tikzcd}
\tag{4.9}
$$

establishing a substantial *symmetry* between the non-holonomic Lagrangian set-up and the non-holonomic Hamiltonian one.

The situation is made more transparent by removing any hierarchy between $\mathcal{A}$ and $\mathcal{S}$, and considering both of them on a perfectly equal footing as submanifolds $i_{\mathcal{A}} : \mathcal{A} \to j_1(\mathcal{V}_{n+1})$, $i_{\mathcal{S}} : \mathcal{S} \to \Pi(\mathcal{V}_{n+1})$, mutually related by a diffeomorphism $\ell : \mathcal{A} \to \mathcal{S}$. By analogy with the holonomic case, we shall call ℓ the *Legendre transformation*.

The Lagrangian scenario is then recovered by regarding $\mathcal{A}$ as the primary geometrical environment, and summarizing both $\mathcal{S}$ and ℓ in the *Legendre map* $\lambda := i_{\mathcal{S}} \cdot \ell : \mathcal{A} \to \Pi(\mathcal{V}_{n+1})$. The outline is completed by adopting a 'fully Lagrangian' picture of the map λ, namely relating it to the assignment of a *hyperregular contact connection* over $\mathcal{A}$.

In a perfectly symmetric way, we may envisage a *Hamiltonian scenario*, based on the choice of $\mathcal{S}$ as the primary space. The missing information (the submanifold $\mathcal{A}$ and the Legendre transformation ℓ) is then summarized in an embedding $\kappa := i_{\mathcal{A}} \cdot \ell^{-1} : \mathcal{S} \to j_1(\mathcal{V}_{n+1})$, henceforth called the *Legendre inverse map*. To complete the set-up we now have to look for a 'fully Hamiltonian' picture of κ, relating it to a suitable geometrical object over the submanifold $\mathcal{S}$.

To this end, let us denote by $(V(\mathcal{S}))^0$ the bundle of semibasic 1-forms over $\mathcal{S}$, identified with the annihilator of the vertical bundle over $\mathcal{S}$.

Definition 4.2. *A linear functional* $\mathfrak{T}$ *on* $(V(\mathcal{S}))^0$ *satisfying* $\mathfrak{T}(\mathrm{d}t) \equiv 1$ *will be called a* dynamical scheme *on* $\mathcal{S}$*. The associated kernel* $\ker(\mathfrak{T}) \subset (V(\mathcal{S}))^0$*, henceforth denoted by* $C(\mathcal{S})$*, will be called a* contact co-distribution *over* $\mathcal{S}$*. Every section* $\sigma : \mathcal{S} \to C(\mathcal{S})$ *will be called a contact* 1*-form over* $\mathcal{S}$*.*

Given any dynamical scheme $\mathfrak{T}$ on $\mathcal{S}$, a straightforward argument shows that the totality of vectors $X \in T(\mathcal{S})$ satisfying the condition

$$\langle X, \sigma \rangle = \mathfrak{T}(\sigma) \qquad \forall \sigma \in (V(\mathcal{S}))^0 \tag{4.10}$$

forms an affine subbundle $\tau(\mathcal{S}) \subset T(\mathcal{S})$, modelled on the vertical bundle $V(\mathcal{S})$. We shall call the latter the *dynamical bundle* associated with the scheme $\mathfrak{T}$ on $\mathcal{S}$. Every section $Z : \mathcal{S} \to \tau(\mathcal{S})$ will be called a *dynamical flow* over $\mathcal{S}$.

We let the reader verify that the assignment of the dynamical bundle $\tau(\mathcal{S})$ is mathematically equivalent to the assignment of the dynamical scheme $\mathfrak{T}$, and that every pair of dynamical flows $Z, Z' : \mathcal{S} \to \tau(\mathcal{S})$ differs by a vertical vector field.

The relationship between definition 4.2 and the Legendre inverse map is clarified by the following:

Proposition 4.2. *Every differentiable map $\kappa : \mathcal{S} \to j_1(\mathcal{V}_{n+1})$ fibred over $\mathcal{V}_{n+1}$ determines a dynamical scheme $\mathfrak{T}$ on $\mathcal{S}$, uniquely defined by the requirement*

$$\mathfrak{T}(\mathrm{d}q^i) = \kappa^*(\dot{q}^i). \tag{4.11}$$

The contact co-distribution $C(\mathcal{S})$ associated with $\mathfrak{T}$ coincides with the pull-back $\kappa^*(C(j_1(\mathcal{V}_{n+1})))$ of the contact bundle over $j_1(\mathcal{V}_{n+1})$. Conversely, knowledge of $\mathfrak{T}$ is mathematically equivalent to knowledge of the map κ.

The proof is entirely straightforward, and is left to the reader.

For further use, we observe that the assignment of a dynamical scheme $\mathfrak{T}$ determines not only a corresponding contact co-distribution $C(\mathcal{S})$, but also a *bilinear pairing* between vertical vectors and contact 1-forms at each point $\varsigma \in \mathcal{S}$. Indeed, denoting by κ the map associated with $\mathfrak{T}$ through equation (4.11), it is easily recognized that:

- the push-forward $(\kappa_\varsigma)_*$ maps the vertical space $V_\varsigma(\mathcal{S})$ into (a subspace of) the vertical space $V_{\kappa(\varsigma)}(j_1(\mathcal{V}_{n+1}))$;
- the pull-back $(\kappa_\varsigma)_*{}^*$ sets up a one-to-one correspondence between the vector spaces $C_{\kappa(\varsigma)}(j_1(\mathcal{V}_{n+1}))$ and $C_\varsigma(\mathcal{S})$.

For any $V \in V_\varsigma(\mathcal{S}), \sigma \in C_\varsigma(\mathcal{S})$, the required pairing is then expressed in terms of the analogous pairing (2.3) in $j_1(\mathcal{V}_{n+1})$ through the prescription

$$\langle V \| \sigma \rangle := \langle \kappa_*(V) \| \hat{\sigma} \rangle \tag{4.12}$$

with $\hat{\sigma} \in C_{\kappa(\varsigma)}(j_1(\mathcal{V}_{n+1}))$ uniquely defined by the requirement $\sigma = (\kappa_\varsigma)_*{}^*(\hat{\sigma})$.

Definition 4.3. *The annihilator of the vertical space $V(\mathcal{S})$ under the pairing (4.12), henceforth denoted by $\chi(\mathcal{S})$, will be called the* Chetaev bundle *over $\mathcal{S}$ induced by the dynamical scheme $\mathfrak{T}$.*

In coordinates, the previous arguments are formalized by adopting a representation of the submanifold $\mathcal{S}$ of the form

$$p_i = \varphi_i(t, q^1, \ldots, q^n, \mu_1, \ldots, \mu_r). \tag{4.13}$$

This provides the local representations

$$\hat{\Theta} = \mathrm{d}u - p_0\,\mathrm{d}t - \varphi_i(t, q^1, \ldots, q^n, \mu_1, \ldots, \mu_r)\,\mathrm{d}q^i \tag{4.14a}$$

for the non-holonomic Liouville 1-form, and

$$\hat{\Omega} = \mathrm{d}p_0 \wedge \mathrm{d}t + \mathrm{d}\varphi_i \wedge \mathrm{d}q^i \tag{4.14b}$$

for the associated curvature 2-form.

The Legendre transformation is described locally by the system

$$\mu_A = \mu_A(t, q^1, \ldots, q^n, z^1, \ldots, z^r) \tag{4.15a}$$

with inverse

$$z^A = z^A(t, q^1, \ldots, q^n, \mu_1, \ldots, \mu_r). \tag{4.15b}$$

Equations (4.13) and (4.15*a*), together, yield back the Legendre map (4.7), and therefore also the components of the contact connection $\hat{\vartheta}$. Conversely, equations (4.15*b*), together with the representation (2.8*a*) of the submanifold $\mathcal{A}$, provide a description of the Legendre inverse map in terms of the variables t, q^i, μ_A in the local form

$$\dot{q}^i = \psi^i(t, q^1, \ldots, q^n, \mu_1, \ldots, \mu_r) \tag{4.16}$$

with $\psi^i(t, q^i, \mu_A) := \psi^i(t, q^i, z^A(t, q^i, \mu_A))$.

The dynamical scheme determined by map (4.16) is described by the equations

$$\mathfrak{T}(\mathrm{d}q^i) = \psi^i(t, q^1, \ldots, q^n, \mu_1, \ldots, \mu_r) \qquad \mathfrak{T}(\mathrm{d}t) = 1 \tag{4.17a}$$

while the associated contact co-distribution is locally spanned by the 1-forms,

$$\tilde{\omega}^i := \mathrm{d}q^i - \mathfrak{T}(\mathrm{d}q^i)\,\mathrm{d}t = \mathrm{d}q^i - \psi^i(t, q^i, \mu_A)\,\mathrm{d}t. \tag{4.17b}$$

The dynamical flows associated with $\mathfrak{T}$ have local expressions of the form

$$Z = \frac{\partial}{\partial t} + \psi^i \frac{\partial}{\partial q^i} + Z_A \frac{\partial}{\partial \mu_A} \tag{4.18}$$

for arbitrary choice of the components $Z_A(t, q^i, \mu_A)$. For any $V = V_A \frac{\partial}{\partial \mu_A} \in V(\mathcal{S})$, $\sigma = \sigma_i \tilde{\omega}^i \in C(\mathcal{S})$, the pairing (4.12) takes the explicit form

$$\langle V \| \sigma \rangle = \left\langle V_A \frac{\partial \psi^j}{\partial \mu_A} \frac{\partial}{\partial \dot{q}^j} \middle\| \sigma_i \tilde{\omega}^i \right\rangle = \sigma_i V_A \frac{\partial \psi^i}{\partial \mu_A}. \tag{4.19}$$

In particular, a contact 1-form $\nu = \nu_i \tilde{\omega}^i$ belongs to the Chetaev bundle $\chi(\mathcal{S})$ induced by $\mathfrak{T}$ if and only if the components ν_i satisfy the *Chetaev conditions*

$$\nu_i \frac{\partial \psi^i}{\partial \mu_A} = 0. \tag{4.20}$$

4.2. Hamiltonian pairs

We now discuss the dynamical implications of the concepts introduced so far. To start with, we consider the situation arising from the assignment of a hyperregular Lagrangian pair $(l, \hat{\vartheta})$ on $\mathcal{A}$. In this case, proceeding as in section 4.1, we can use the connection $\hat{\vartheta}$ to set up the bundle homomorphism (4.4) and the associated structures (non-holonomic Hamiltonian bundles, Legendre map, etc).

By means of the correspondence (4.6), the Lagrangian section $l : \mathcal{A} \to \mathcal{L}(\mathcal{A})$ may be lifted to an embedding $\widetilde{\Lambda} \cdot l : \mathcal{A} \to \mathcal{H}(\mathcal{V}_{n+1})$, described in coordinates as

$$t = t \qquad q^i = q^i \qquad p_0 = L(t, q^i, z^A) - \varphi_i \psi^i \qquad p_i = \varphi_i(t, q^i, z^A). \tag{4.21}$$

By equation (4.21) it is easily seen that, whenever two Lagrangian pairs $(l, \hat{\vartheta})$, $(l', \hat{\vartheta}')$ are *gauge equivalent*, i.e. when they differ by a translation

$$l' = l + \frac{\partial f}{\partial t} + \frac{\partial f}{\partial q^i}\psi^i \qquad \hat{\vartheta}' = \hat{\vartheta} - \frac{\partial f}{\partial q^i}\tilde{\omega}^i$$

with $f = f(t, q^i) \in \mathcal{F}(\mathcal{V}_{n+1})$, the corresponding sections $\widetilde{\Lambda} \cdot l$, $\widetilde{\Lambda}' \cdot l'$ are similarly related by the action of the gauge group on $\mathcal{H}(\mathcal{V}_{n+1})$, namely

$$p_0 \cdot \widetilde{\Lambda}' \cdot l' = p_0 \cdot \widetilde{\Lambda} \cdot l + \frac{\partial f}{\partial t} \qquad p_i \cdot \widetilde{\Lambda}' \cdot l' = p_i \cdot \widetilde{\Lambda} \cdot l + \frac{\partial f}{\partial q^i}.$$

Let Σ denote the image space $\widetilde{\Lambda} \cdot l(\mathcal{A}) \subset \mathcal{H}(\mathcal{V}_{n+1})$. In view of definition 4.1, in the hyperregular case, Σ is a submanifold of co-dimension 1 of the non-holonomic Hamiltonian bundle $\mathcal{H}(\mathcal{S})$, fibred over $\mathcal{S}$. Accordingly, there exists a unique section $h : \mathcal{S} \to \mathcal{H}(\mathcal{S})$ giving rise to the commutative diagram

$$\begin{tikzcd} \mathcal{L}(\mathcal{A}) \arrow[r, "\widetilde{\Lambda}"] & \mathcal{H}(\mathcal{S}) \\ \mathcal{A} \arrow[u, "l"] \arrow[r, "\lambda"] & \mathcal{S} \arrow[u, "h"'] \end{tikzcd} . \tag{4.22}$$

We shall call h the (non-holonomic) *Hamiltonian section*. Setting

$$H := \varphi_i \psi^i - L \tag{4.23}$$

and referring the manifolds $\mathcal{S}$ and $\mathcal{H}(\mathcal{S})$ respectively to local coordinates t, q^i, μ_A and t, q^i, μ_A, p_0, equations (4.15*b*) and (4.21) provide a representation of h in the 'Cartesian' form

$$p_0 + H(t, q^i, \mu_A) = 0 \tag{4.24}$$

while the consistency relations (3.14) are transformed into

$$\frac{\partial H}{\partial \mu_A} - \psi^i \frac{\partial \varphi_i}{\partial \mu_A} = \varphi_i \frac{\partial \psi^i}{\partial \mu_A} - \frac{\partial L}{\partial \mu_A} = \left(\varphi_i \frac{\partial \psi^i}{\partial z^B} - \frac{\partial L}{\partial z^B}\right) \frac{\partial z^B}{\partial \mu_A} = 0. \tag{4.25}$$

The previous results may be reformulated strictly in Hamiltonian terms. For this purpose, let us consider the geometrical set-up determined by the simultaneous assignment of a section $h : \mathcal{S} \to \mathcal{H}(\mathcal{S})$, expressed locally as $p_0 = -H(t, q^i, \mu_A)$, and of a dynamical scheme $\mathfrak{T}$, described locally by the equations $\mathfrak{T}(\mathrm{d}q^i) = \psi^i(t, q^i, \mu_A)$. By means of h, the curvature 2-form (4.14*b*) associated with the non-holonomic Liouville connection $\hat{\Theta}$ may be pulled back to $\mathcal{S}$. The resulting expression

$$\Omega := h^*(\hat{\Omega}) = -\mathrm{d}H \wedge \mathrm{d}t + \mathrm{d}\varphi_i \wedge \mathrm{d}q^i \tag{4.26}$$

will be called the (non-holonomic) *Poincaré–Cartan* 2*-form* of h.

Definition 4.4. *A Hamiltonian section $h : \mathcal{S} \to \mathcal{H}(\mathcal{S})$ and a dynamical scheme $\mathfrak{T}$ are said to form a Hamiltonian* pair *if and only if the correspondence $X \to X \,\lrcorner\, \Omega$ determined by the Poincaré–Cartan* 2*-form (4.26) maps vertical vectors into contact* 1*-forms, i.e. if and only if the relation*

$$\mathfrak{T}(V \,\lrcorner\, \Omega) \equiv 0 \tag{4.27}$$

holds identically for any $V \in V(\mathcal{S})$.

In local coordinates, setting $V = V_A \frac{\partial}{\partial \mu_A}$, and preserving the notation (4.17*b*) for the 1-forms $\tilde{\omega}^i$, we have the explicit expression

$$V \,\lrcorner\, \Omega = V_A \left(-\frac{\partial H}{\partial \mu_A}\,\mathrm{d}t + \frac{\partial \varphi_i}{\partial \mu_A}\,\mathrm{d}q^i\right) = V_A \left[\left(-\frac{\partial H}{\partial \mu_A} + \psi^i \frac{\partial \varphi_i}{\partial \mu_A}\right)\mathrm{d}t + \frac{\partial \varphi_i}{\partial \mu_A}\tilde{\omega}^i\right].$$

The requirement (4.27) is therefore equivalent to the validity of the conditions

$$\frac{\partial H}{\partial \mu_A} = \psi^i \frac{\partial \varphi_i}{\partial \mu_A} \qquad A = 1, \ldots, r \tag{4.28a}$$

identical to equations (4.25). They, in turn, imply the local representation

$$V \rfloor \Omega = V_A \frac{\partial \varphi_i}{\partial \mu_A} \tilde{\omega}^i \qquad \forall V \in (V(\mathcal{S}))^0. \tag{4.28b}$$

A comparison of equation (4.28*b*) with equation (4.19) shows that every Hamiltonian pair determines a *scalar product* between vertical vectors, based on the prescription

$$(X, Y) := \langle X \| Y \rfloor \Omega \rangle = \left\langle X \,\middle\|\, Y_B \frac{\partial \varphi_i}{\partial \mu_B} \tilde{\omega}^i \right\rangle = X_A Y_B \frac{\partial \psi^i}{\partial \mu_A} \frac{\partial \varphi_i}{\partial \mu_B} \tag{4.29}$$

(see the analogous expression (3.16), valid in the Lagrangian context). Setting

$$K^{AB} := \left(\frac{\partial}{\partial \mu_A}, \frac{\partial}{\partial \mu_B} \right) = \frac{\partial \psi^i}{\partial \mu_A} \frac{\partial \varphi_i}{\partial \mu_B} \tag{4.30}$$

equation (4.28*a*) implies the symmetry relations

$$K^{AB} = \frac{\partial^2 H}{\partial \mu_A \partial \mu_B} - \psi^i \frac{\partial^2 \varphi_i}{\partial \mu_A \partial \mu_B} = K^{BA} \tag{4.31}$$

mathematically equivalent to $(X, Y) = (Y, X)$. Note that, by construction, the scalar product (4.29) depends on the dynamical scheme $\mathfrak{T}$, but is independent of the section h. Conversely, we have the following:

Proposition 4.3. *A dynamical scheme $\mathfrak{T}$ may be completed (locally) to a Hamiltonian pair $(h, \mathfrak{T})$ if and only if the functions $\psi^i(t, q^i, \mu_A) := \mathfrak{T}(\mathrm{d}q^i)$ involved in the representation of $\mathfrak{T}$ satisfy the symmetry requirement*

$$\frac{\partial \varphi_i}{\partial \mu_A} \frac{\partial \psi^i}{\partial \mu_B} = \frac{\partial \varphi_i}{\partial \mu_B} \frac{\partial \psi^i}{\partial \mu_A}. \tag{4.32}$$

Under the stated assumption, the section h is determined up to an arbitrary transformation $h \to h + \varrho(t, q^i)$.

The proof is identical to the proof of proposition 3.1 and will be omitted.

In analogy with the Lagrangian case, a Hamiltonian pair $(h, \mathfrak{T})$ will be called *regular* if and only if the associated scalar product (4.29) is non-degenerate, i.e. if and only if the matrix (4.30) is everywhere non-singular on $\mathcal{S}$. It will be called hyperregular if, in addition, the map $\kappa : \mathcal{S} \to j_1(\mathcal{V}_{n+1})$ induced by $\mathfrak{T}$ is an *embedding*. A dynamical scheme $\mathfrak{T}$ will be called regular (hyperregular) if and only if it may be completed locally to a regular (hyperregular) Hamiltonian pair. Recalling definition 4.3, we have then the following:

Proposition 4.4. *The contact co-distribution associated with any regular dynamical scheme $\mathfrak{T}$ splits into the direct sum*

$$C(\mathcal{S}) = \chi(\mathcal{S}) \oplus_{\mathcal{S}} g(V(\mathcal{S})) \tag{4.33}$$

where $\chi(\mathcal{S})$ and $g(V(\mathcal{S}))$ denote respectively the Chetaev bundle induced by $\mathfrak{T}$ and the image of the vertical space $V(\mathcal{S})$ under the map (4.28b).

The proof is a replica of the one given for proposition 3.2, and will be omitted.

We finally discuss the role of the regular Hamiltonian pairs in the representation of *dynamical flows* over $\mathcal{S}$, meant as sections $Z : \mathcal{S} \to \tau(\mathcal{S})$ of the dynamical bundle. Once again, the argument is essentially a restatement of the analogous result established in section 3.2: given any regular Hamiltonian pair $(h, \mathfrak{T})$ we consider the exterior ideal generated by the Poincaré–Cartan 2-form (4.26) and by the Chetaev 1-forms, and denote by $\mathcal{D}$ the associated characteristic distribution,

$$\mathcal{D} := \{X | X \in T(\mathcal{S}),\, X \,\lrcorner\, \Omega \in \chi(\mathcal{S}),\, X \,\lrcorner\, \chi(\mathcal{S}) = \{0\}\}. \tag{4.34}$$

We have then the following:

Theorem 4.1. *Given any regular Lagrangian pair* $(h, \mathfrak{T})$*, the characteristic distribution (4.34) is one-dimensional, and coincides with the linear span of a dynamical flow* Z *over* $\mathcal{S}$*, uniquely determined by the equation*

$$\left(Z(\varphi_i) + \frac{\partial H}{\partial q^i} - \psi^k \frac{\partial \varphi_k}{\partial q^i}\right) \frac{\partial \psi^i}{\partial \mu_A} = 0. \tag{4.35}$$

Proof. A straightforward check shows that the most general vector X satisfying $X \,\lrcorner\, \nu = 0$, $\forall \nu \in \chi(\mathcal{S})$ is necessarily of the form

$$X = X^0 \left(\frac{\partial}{\partial t} + \psi^i \frac{\partial}{\partial q^i}\right) + X_A \frac{\partial \psi^i}{\partial \mu_A} \frac{\partial}{\partial q^i} + V_A \frac{\partial}{\partial \mu_A} \tag{4.36}$$

for arbitrary choice of X^0, X_A and V_A. At the same time, taking equations (4.17*b*) and (4.28) into account, the Poincaré–Cartan 2-form (4.26) may be written locally as

$$\Omega = (-\mathrm{d}H + \psi^k \,\mathrm{d}\varphi_k) \wedge \mathrm{d}t + \mathrm{d}\varphi_i \wedge \tilde{\omega}^i = \left[\left(\frac{\partial H}{\partial q^i} - \psi^k \frac{\partial \varphi_k}{\partial q^i}\right) \mathrm{d}t + \mathrm{d}\varphi_i\right] \wedge \tilde{\omega}^i. \tag{4.37}$$

In view of equation (4.37), the characteristic distribution (4.34) consists of the totality of vectors of the form (4.36) satisfying the condition

$$\left(\frac{\partial H}{\partial q^i} - \psi^k \frac{\partial \varphi_k}{\partial q^i}\right) \left(X^0 \tilde{\omega}^i - X_A \frac{\partial \psi^i}{\partial \mu_A} \mathrm{d}t\right) + X(\varphi_i) \tilde{\omega}^i - X_A \frac{\partial \psi^i}{\partial \mu_A} \mathrm{d}\varphi_i \in \chi(\mathcal{S}).$$

From this, taking equations (4.20) and (4.30) into account, and proceeding as in section 3.2, it may be readily seen that, under the regularity assumption $\det K^{AB} \neq 0$, the required distribution is spanned by a single vector field

$$Z = \frac{\partial}{\partial t} + \psi^i \frac{\partial}{\partial q^i} + Z_A \frac{\partial}{\partial \mu_A} \tag{4.38}$$

uniquely determined by the requirement (4.35). Comparison with equation (4.18) shows that Z is a dynamical flow over $\mathcal{S}$. □

According to theorem 4.1, every regular Lagrangian pair $(h, \mathfrak{T})$ gives rise to a well-posed problem of motion through the associated characteristic distribution (4.34). The equations of evolution for the unknowns $q^i(t)$, $\mu_A(t)$ are easily obtained from equations (4.35) and (4.38). The details are left to the reader.

4.3. Canonical frameworks

The symmetry between the Lagrangian and the Hamiltonian approaches is further enhanced through the adoption of an 'intermediate' set-up, consisting of the simultaneous and independent assignment of *both* submanifolds $\mathcal{A}$ and $\mathcal{S}$.

Definition 4.5. *A pair of mutually diffeomorphic submanifolds* $i_{\mathcal{A}} : \mathcal{A} \to j_1(\mathcal{V}_{n+1})$, $i_{\mathcal{S}} : \mathcal{S} \to \Pi(\mathcal{V}_{n+1})$ *will be called a* canonical framework *over* $\mathcal{V}_{n+1}$*. In a given canonical framework* $(\mathcal{A}, \mathcal{S})$*:*

- *a contact connection $\hat{\vartheta}$ over $\mathcal{A}$ is called admissible if and only if the associated map $\lambda : \mathcal{A} \to \Pi(\mathcal{V}_{n+1})$ satisfies $\lambda(\mathcal{A}) \subset \mathcal{S}$;*
- *a dynamical scheme $\mathfrak{T}$ on $\mathcal{S}$ is called admissible if and only if the associated map $\kappa : \mathcal{S} \to j_1(\mathcal{V}_{n+1})$ satisfies $\kappa(\mathcal{S}) \subset \mathcal{A}$;*
- *a Lagrangian pair $(l, \hat{\vartheta})$ (respectively, a Hamiltonian pair $(h, \mathfrak{T})$) is called admissible if and only if the contact connection $\hat{\vartheta}$ (respectively, the dynamical scheme $\mathfrak{T}$) is admissible.*

In view of definition 4.5, every admissible contact connection $\hat{\vartheta}$ is uniquely determined by a corresponding differentiable map $\hat{\lambda} : \mathcal{A} \to \mathcal{S}$, related to the Legendre map λ by the factorization $\lambda = i_{\mathcal{S}} \cdot \hat{\lambda}$. In a similar way, every admissible dynamical scheme $\mathfrak{T}$ is determined by the map $\hat{\kappa} : \mathcal{S} \to \mathcal{A}$ involved in the factorization $\kappa = i_{\mathcal{A}} \cdot \hat{\kappa}$. Through the graphs of the associated maps, the admissible contact connections and the admissible dynamical schemes may therefore be identified with *submanifolds* of the product bundle $\mathcal{A} \times_{\mathcal{V}_{n+1}} \mathcal{S}$, projecting diffeomorphically onto A and $\mathcal{S}$ respectively.

The previous arguments help in overcoming the redundancy implicit in the concepts of Lagrangian and Hamiltonian pairs, i.e. in a formalism based on the use of *non-independent* geometrical objects, related to each other by *consistency conditions*. This aspect is clarified by the following:

Proposition 4.5. *Consider a canonical framework $(\mathcal{A}, \mathcal{S})$ over $\mathcal{V}_{n+1}$, described locally by the equations*

$$\dot{q}^i = \psi^i(t, q^1, \ldots, q^n, z^1, \ldots, z^r) \tag{4.39a}$$

$$p_i = \varphi_i(t, q^1, \ldots, q^n, \mu_1, \ldots, \mu_r). \tag{4.39b}$$

In the product bundle $\mathcal{A} \times_{\mathcal{V}_{n+1}} \mathcal{S}$ consider the matrix function

$$M^A{}_B := \frac{\partial \varphi_i}{\partial \mu_A} \frac{\partial \psi^i}{\partial z^B}. \tag{4.40}$$

Then:

(a) A section $l : \mathcal{A} \to \mathcal{L}(\mathcal{A})$, with local equation $\dot{u} = L(t, q^1, \ldots, q^n, z^1, \ldots, z^r)$, may be completed to a unique admissible Lagrangian pair $(l, \hat{\vartheta})$ if and only if the subset $\mathfrak{G} \subset \mathcal{A} \times_{\mathcal{V}_{n+1}} \mathcal{S}$ defined implicitly by the system

$$\frac{\partial L}{\partial z^A} = \varphi_i \frac{\partial \psi^i}{\partial z^A} \tag{4.41a}$$

is a section of the bundle $\mathcal{A} \times_{\mathcal{V}_{n+1}} \mathcal{S} \to \mathcal{A}$, i.e. if and only if the matrix (4.40) is everywhere non-singular on $\mathfrak{G}$. Under the stated circumstance, the pair $(l, \hat{\vartheta})$ is regular if and only if the restriction to the submanifold $\mathfrak{G}$ of the projection $\mathcal{A} \times_{\mathcal{V}_{n+1}} \mathcal{S} \to \mathcal{S}$ is an immersion, i.e. if and only if the matrix

$$\frac{\partial^2 L}{\partial z^A \partial z^B} - \varphi_i \frac{\partial^2 \psi^i}{\partial z^A \partial z^B} \tag{4.41b}$$

is also everywhere non-singular on $\mathfrak{G}$. It is hyperregular if and only if $\mathfrak{G}$ is a section of $\mathcal{A} \times_{\mathcal{V}_{n+1}} \mathcal{S} \to \mathcal{S}$.

(b) A section $h : \mathcal{S} \to \mathcal{H}(\mathcal{S})$, with local equation $p_0 + H(t, q^1, \ldots, q^n, \mu_1, \ldots, \mu_r) = 0$, may be completed to a unique admissible Hamiltonian pair $(h, \mathfrak{T})$ if and only if the subset $\mathfrak{G} \subset \mathcal{A} \times_{\mathcal{V}_{n+1}} \mathcal{S}$ defined implicitly by the system

$$\psi^i \frac{\partial \varphi_i}{\partial \mu_A} = \frac{\partial H}{\partial \mu_A} \tag{4.42a}$$

is a section of the bundle $\mathcal{A} \times_{\mathcal{V}_{n+1}} \mathcal{S} \to \mathcal{S}$, *i.e. if and only if the matrix (4.40) is everywhere non-singular on* $\mathfrak{G}$. *Under the stated circumstance, the pair* $(h, \mathfrak{T})$ *is regular if and only if the restriction to the submanifold* $\mathfrak{G}$ *of the projection* $\mathcal{A} \times_{\mathcal{V}_{n+1}} \mathcal{S} \to \mathcal{A}$ *is an immersion, i.e. if and only if the matrix*

$$\frac{\partial^2 H}{\partial \mu_A \partial \mu_B} - \psi^i \frac{\partial^2 \varphi_i}{\partial \mu_A \partial \mu_B} \tag{4.42b}$$

is also everywhere non-singular on $\mathfrak{G}$. *It is hyperregular if and only if* $\mathfrak{G}$ *is a section of* $\mathcal{A} \times_{\mathcal{V}_{n+1}} \mathcal{S} \to \mathcal{A}$.

Proof. As pointed out above, every admissible contact connection $\hat{\vartheta}$ over $\mathcal{A}$ is identified by a corresponding map $\mathcal{A} \to \mathcal{S}$, expressed locally as $\mu_A = \mu_A(t, q^i, z^A)$. At the same time, equations (4.41*a*) are precisely the consistency conditions required in order for $(l, \hat{\vartheta})$ to form a Lagrangian pair. Finally, as long as the matrix (4.40) is non-singular along $\mathfrak{G}$ equations (4.41*a*) may be solved uniquely for the μ_A as functions of t, q^i, z^A by the implicit function theorem.

Assertion (a) follows from these facts and from the definition of regularity and hyperregularity given in section 3.2. The proof of assertion (b) is analogous, and will be omitted. □

Proposition 4.5 shows that, in a canonical framework, the assignment of either a Lagrangian or a Hamiltonian section subject to suitable regularity requirements is sufficient to determine a dynamical flow, thereby restoring a state of affairs analogous to the one occurring in holonomic mechanics.

From an aesthetic viewpoint, this is especially worthy in the *Hamiltonian* context: prescribing a canonical framework and a section $h : \mathcal{S} \to \mathcal{H}(\mathcal{S})$ is in fact equivalent to assigning two mutually diffeomorphic submanifolds $\mathcal{A} \subset j_1(\mathcal{V}_{n+1})$ and $\Sigma := h(\mathcal{S}) \subset \mathcal{H}(\mathcal{S})$, respectively providing a gauge-independent description of the *kinematics* of the system, and a gauge-dependent description of its *dynamics*.

The details are straightforward, and are left to the reader.

Appendix A. Free Lagrangians and constrained dynamics

The following arguments emphasize the difference between *extrinsic* Lagrangians, meant as tools for the representation of *constrained dynamics* in the sense described in section 3.3, and *free* (or unconstrained) Lagrangians, expressing how the evolution of the system would look *in the absence of constraints*. The first example illustrates the situation in *holonomic* mechanics.

Example A.1. Particle of unit mass, subject to the action of a constant force $\boldsymbol{F}$. In Cartesian coordinates, the evolution is described by the dynamical flow

$$\tilde{Z} = \frac{\partial}{\partial t} + \dot{x}^i \frac{\partial}{\partial x^i} + F^i \frac{\partial}{\partial \dot{x}^i}. \tag{A.1}$$

Without the loss of generality, we may assume $\boldsymbol{F} = F\boldsymbol{e}_3 \Rightarrow F^i = F\delta^i_3$.

Given any constant, symmetric, non-singular matrix a_{ij}, a straightforward check shows that the function

$$\tilde{L} = a_{hk}\left(\tfrac{1}{2}\dot{x}^h\dot{x}^k + F^h x^k\right) = \tfrac{1}{2}a_{hk}\dot{x}^h\dot{x}^k + a_{3k}Fx^k \tag{A.2}$$

provides a possible Lagrangian for the flow (A.1). In addition to the ‘natural’ choice $\tilde{L} = T + U$, corresponding to $a_{ij} = \delta_{ij}$, we have therefore a whole class of gauge-inequivalent alternatives, all giving rise to the same equations of motion.

Let us now restrict the mobility of the system by the *positional* constraint

$$x^3 = 0.$$

The pull-back of the Lagrangian (A.2) to the newer configuration spacetime is

$$L = \frac{1}{2}\sum_{A,B=1}^{2} a_{AB}\dot{x}^A\dot{x}^B + \sum_{A=1}^{2} a_{3A}Fx^A.$$

This gives rise to the Lagrange equations

$$\sum_{B=1}^{2} a_{AB}\ddot{x}^B - a_{3A}F = 0$$

which, in general, *do not* imply $\ddot{x}^B = 0$, and therefore do not express the dynamical behaviour of the constrained system, except for special choices of the matrix a_{ij} (including, among others, the 'natural' choice $a_{ij} = \delta_{ij}$).

Therefore, even in the holonomic context, pulling back a correct 'unconstrained' Lagrangian may fail to produce a Lagrangian for the constrained dynamical flow.

Example A.2. Particle of unit mass, moving in a plane, under the action of a non-conservative force

$$\boldsymbol{F} = -kx(\boldsymbol{e}_1 + \boldsymbol{e}_2) \qquad k = \text{const}$$

and subject to the non-integrable ideal kinetic constraint

$$\dot{y} = \dot{x} + \beta x.$$

The unconstrained dynamical flow is

$$\tilde{Z} = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + \dot{y}\frac{\partial}{\partial y} - kx\left(\frac{\partial}{\partial \dot{x}} + \frac{\partial}{\partial \dot{y}}\right). \tag{A.3}$$

The constrained dynamical flow Z is obtained projecting $\tilde{Z}$ on the submanifold $\mathcal{A} \subset j_1(\mathcal{V}_{n+1})$ by means of the fibre metric induced by the kinetic energy $T = \frac{1}{2}(\dot{x}^2 + \dot{y}^2)$ [1, 3]. Referring $\mathcal{A}$ to (global) coordinates $t, x, y, \dot{x}$, the result is

$$Z = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + (\dot{x} + \beta x)\frac{\partial}{\partial y} - \left(\frac{\beta}{2}\dot{x} + kx\right)\frac{\partial}{\partial \dot{x}}. \tag{A.4}$$

A straightforward check shows that the function

$$\tilde{L} = \tfrac{1}{2}[\dot{x}^2 + (\dot{x} - \dot{y})^2] - \tfrac{1}{2}kx^2 \tag{A.5}$$

is a Lagrangian for the flow (A.3), but is *not* an extrinsic Lagrangian for the constrained flow (A.4). The first assertion is obvious. To prove the second one, we consider the Poincaré–Cartan 1-form of $\tilde{L}$,

$$\tilde{\vartheta} = \tilde{L}\,\mathrm{d}t + \frac{\partial \tilde{L}}{\partial \dot{q}^k}\omega^k$$

and pull it back to the submanifold $\mathcal{A} \subset j_1(\mathcal{V}_{n+1})$. We then evaluate the dynamical flow on $\mathcal{A}$ resulting from the algorithm described in theorem 3.1, namely the vector field $X \in \mathcal{D}^1(\mathcal{A})$ defined by the conditions

$$X \,\lrcorner\, i^*(\mathrm{d}\tilde{\vartheta}) \in \chi(\mathcal{A}) \qquad X \,\lrcorner\, \chi(\mathcal{A}) = \{0\}, \qquad \langle X, \mathrm{d}t\rangle = 1 \tag{A.6}$$

where $\chi(\mathcal{A})$ denotes the Chetaev bundle over $\mathcal{A}$. By direct calculation, we get the result

$$X = \frac{\partial}{\partial t} + \dot{x}\frac{\partial}{\partial x} + (\dot{x} + \beta x)\frac{\partial}{\partial y} - kx\frac{\partial}{\partial \dot{x}}$$

clearly different from the flow (A.4). Once more, this shows that, even though the Lagrangian (A.5) carries full information on the evolution of the unconstrained system, it does not yield the right equations of motion for the constrained one.

In spite of this fact, the constrained flow *is* derivable from an intrinsic Lagrangian pair $\dot{u} = L(t, x, y, \dot{x})$, $\hat{\vartheta} = \tilde{\omega}^0 + \varphi_i\,\tilde{\omega}^i$ through equation (3.23). A straightforward check shows that the ansatz

$$L = \dot{x}^2 + \frac{2k}{\beta}x\dot{x} \qquad \varphi_1 = \dot{x} \quad \varphi_2 = \dot{x} + \frac{2k}{\beta}x \tag{A.7}$$

does the job. Of course, on the basis of theorem 3.2, one may then work backwards, and construct an *ad hoc* extrinsic Lagrangian, obviously *different* from the unconstrained one, reproducing the pair (A.7) through the pull-back algorithm. A possible choice is

$$\hat{L} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \left(\frac{2k}{\beta} - \beta\right)x\dot{y} + \left(\frac{\beta^2}{2} - 2k\right)x^2.$$

Once again, the calculations are entirely straightforward, and are left to the reader.

Example A.3. Disc of radius R and mass m, rolling without sliding on a horizontal plane, and subject to 'gyroscopic' interactions, i.e. to velocity-dependent forces satisfying $\sum \underline{F}_i \cdot \underline{v}_i \equiv 0$.

In terms of the Lagrangian coordinates: $q^1 = x = x_G, q^2 = y = y_G$ (G = centre of the disc), $q^3 = \theta$ = angle between the plane of the disc and the xz plane, $q^4 = \varphi$ = proper rotation of the disc, the submanifold $\mathcal{A} \to j_1(\mathcal{V}_{n+1})$ of admissible kinetic states is described by the equations $\dot{q}^i = \psi^i_A z^A$, with

$$\begin{pmatrix} z^1 \\ z^1 \end{pmatrix} = \begin{pmatrix} \dot{\theta} \\ \dot{\varphi} \end{pmatrix} \qquad \psi^i_A = \begin{pmatrix} 0 & -R\cos\theta \\ 0 & -R\sin\theta \\ 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{A.8}$$

The pull-back of the unconstrained kinetic Poincaré–Cartan 1-form is

$$T\,\mathrm{d}t + p_i\tilde{\omega}^i := i^*\left(\tilde{T}\,\mathrm{d}t + \frac{\partial\tilde{T}}{\partial\dot{q}^i}\omega^i\right)$$

where $\tilde{T} = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{8}mR^2(\dot{\theta}^2 + 2\dot{\varphi}^2)$ denotes the unconstrained kinetic energy.

The Lagrange–Chetaev equations of motion read

$$i^*\left[Z\left(\frac{\partial\tilde{T}}{\partial\dot{q}^k}\right) - \frac{\partial\tilde{T}}{\partial q^k} - Q_k\right]\frac{\partial\psi^k}{\partial z^A} = 0$$

mathematically equivalent to

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial z^A} - \frac{\partial T}{\partial q^i}\frac{\partial\psi^i}{\partial z^A} = p_k\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\psi^k}{\partial z^A}\right) - \frac{\partial\psi^k}{\partial q^i}\frac{\partial\psi^i}{\partial z^A}\right] + Q_k\frac{\partial\psi^k}{\partial z^A}. \tag{A.9}$$

Let us now observe the following basic facts:

(i) Given any contact 1-form $\sigma = \sigma_i\tilde{\omega}^i$, a straightforward comparison with equation (A.8) provides the representation

$$\begin{aligned}\sigma_k\left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\psi^k}{\partial z^A}\right) - \frac{\partial\psi^k}{\partial q^i}\frac{\partial\psi^i}{\partial z^A}\right] &= \sigma_k\left[\frac{\partial\psi^i_A}{\partial q^i}\psi^i_B - \frac{\partial\psi^i_B}{\partial q^i}\psi^i_A\right]z^B \\ &= R(\sigma_2\cos\theta - \sigma_1\sin\theta)\varepsilon_{AB}z^B \qquad \text{with} \quad \varepsilon_{AB} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.\end{aligned}$$

(ii) The assumption of gyroscopic forces is reflected in the requirement

$$\sum \underline{F}_i \cdot \frac{\partial \underline{x}_i}{\partial q^k}\dot{q}^k = Q_k \psi^k_A z^A \equiv 0 \quad \forall z^A \quad \Rightarrow \quad Q_k \psi^k_A = C(t,q,z)\varepsilon_{AB} z^B.$$

(iii) If $(l, \hat{\vartheta})$ is a Lagrangian pair and ν is a Chetaev 1-form, $(l, \hat{\vartheta} + \nu)$ is again a Lagrangian pair.

In view of (i) and (ii), taking equation (A.8) into account, one can easily infer the existence of at least one Chetaev 1-form $\nu = \nu_i \tilde{\omega}^i$ satisfying the condition

$$\nu_k \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial \psi^k}{\partial z^A}\right) - \frac{\partial \psi^k}{\partial q^i}\frac{\partial \psi^i}{\partial z^A}\right] = Q_k \frac{\partial \psi^k}{\partial z^A}.$$

The request is in fact equivalent to the system

$$\nu_3 = 0 \qquad \nu_4 = R(\nu_1 \cos\theta + \nu_2 \sin\theta) \qquad C = R(\nu_2 \cos\theta - \nu_1 \sin\theta)$$

for the components ν_i. For definiteness, consider e.g. the solution

$$\nu = \frac{C}{R}\left(-\sin\theta\,\tilde{\omega}^1 + \cos\theta\,\tilde{\omega}^2\right).$$

Collecting all previous results we have then the conclusion: the ansatz

$$L = T = \tfrac{1}{8} m R^2 (\dot{\theta}^2 + 6\dot{\varphi}^2)$$
$$\hat{\vartheta} = p_i \omega^i + \nu = -mR\dot{\varphi}(\cos\theta\,\tilde{\omega}^1 + \sin\theta\,\tilde{\omega}^2) + \tfrac{1}{4} m R^2 (\dot{\theta}\tilde{\omega}^3 + 2\dot{\varphi}\tilde{\omega}^4) + \nu$$

provides a Lagrangian pair for the given dynamical system.

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